Math 1201 **Date:**___________________

3.8 Special Polynomials

Modelling polynomials concretely, pictorially and symbolically, as well as the connection between factoring and multiplication, will continue to be emphasized for the following types of polynomials:

• perfect square trinomials

• difference of squares

Perfect Square Trinomials

A perfect square trinomial is a polynomial resulting from multiplying a binomial by itself. There for it has a has a factored form of $(a+b)^2$. Some examples would be:

$$x^2 + 6x + 9 = (x+3)^2$$

$$x^2 - 8x + 16 = (x-4)^2$$

$$4x^2 + 20x + 25 = (2x+5)^2$$

$$9x^2 - 12x + 4 = (3x-2)^2$$

Can you see any relationship between parameters a, b and c?

We can factor perfect square trinomials the same way we factor any trinomials, namely algebra tiles, product some or decomposition. However if the values of a, b and c are high, it can be very time consuming to find all possible factors. Fortunately there's a test to see if a trinomial is a perfect square. We can then use a simple logical patterns and a simple formula to factor.

Given a trinomial, $ax^2 + bx + c$, ignoring the sign, if $b = 2\sqrt{ac}$, then the trinomial is a perfect square. Then factoring simply becomes the square roots of a and c with the sign of b in the middle. When working with a perfect square trinomial you could rewrite the trinomial $a^2x^2 + 2abx + b^2$ as $(ax)^2 + 2abx + (b)^2$. You can see the factors of the binomial as the square root of the first and last term $(ax+b)(ax+b)$.

Example 1:
In each trinomial, how is the coefficient of the linear term related to the product of the coefficient of the squared term and the constant term?

(A) $4x^2 - 12x + 9$

$\sqrt{4} = 2$ $\quad \sqrt{9} = 3$

$2(2 \times 3) = 12$

$\therefore$ PST

(B) $25x^2 - 20x + 4$

$\sqrt{25} = 5$ $\quad \sqrt{4} = 2$

$2(2 \times 5) = 20$

$\therefore$ PST

(C) $16x^2 + 24x + 9$ $\quad \therefore$ PST

$b = 2\sqrt{a \cdot c}$

$b = 2\sqrt{16 \cdot 9}$

$b = 2\sqrt{144}$

$b = 2 \cdot 12$

$b = 24$

Example 2:
Factor:

(A) $9x^2 + 30x + 25$

$b = 2\sqrt{9 \cdot 25}$

$b = 2\sqrt{225}$

$b = 2 \cdot 15$

$b = 30$ $\quad \therefore$ PST

$(3x+5)(3x+5)$

$= (3x+5)^2$

(B) $4x^2 - 28x + 49$

$b = 2\sqrt{4 \cdot 49}$

$b = 2\sqrt{196}$

$b = 2 \cdot 14$

$b = 28$ $\quad \therefore$ PST

$(2x-7)(2x-7)$

$= (2x-7)^2$

(C) $64x^2 + 144x + 81$

$\sqrt{64} = 8$ $\quad \sqrt{81} = 9$

$2(8 \times 9) = 144$ $\quad \therefore$ PST

$(8x+9)(8x+9)$

$= (8x+9)^2$

(D) $144x^2 - 264x + 121$

$\sqrt{144} = 12$ $\quad \sqrt{121} = 11$

$2(12 \times 11) = 264$ $\quad \therefore$ PST

$(12x-11)(12x-11)$

$= (12x-11)^2$

(E) $361x^2 + 798x + 441$

$b = 2\sqrt{361 \cdot 441}$

$b = 2\sqrt{159201}$

$b = 2 \cdot 399$

$b = 798$ $\quad \therefore$ PST

$(19x+21)(19x+21)$

$= (19x+21)^2$

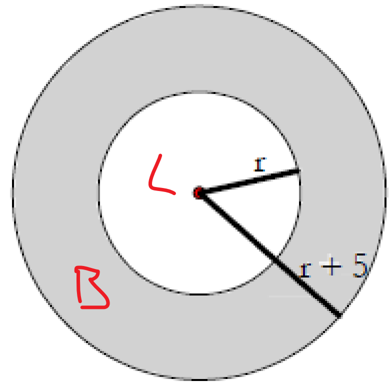
Example 3:
The diagram shows two concentric circles with radii r and $r + 5$. Write an expression for the area of the shaded region and factor the expression completely. If $r = 4$ cm, calculate the area of the shaded region to the nearest tenth of a square centimetre.

$A_o = \pi r^2$

$A_B = \pi(r+5)^2 = \pi(r^2 + 10r + 25)$

$= \pi r^2 + 10\pi r + 25\pi$

$A_L = \pi r^2$



$A_S = A_B - A_L = \cancel{\pi r^2} + 10\pi r + 25\pi - \cancel{\pi r^2}$

$A_S = 10\pi r + 25\pi$ $\quad A = 10\pi(4\text{ cm}) + 25\pi = 204.2\text{ cm}^2$

Example 4:
Determine two values of n that allow the polynomial $25x^2 + nx + 49$ to be a perfect square trinomial. Use them both to factor the trinomial.

$\sqrt{25} = 5$ $\quad 2(5 \times 7)$

$\sqrt{49} = 7$ $\quad = 70$ or -70

Difference of Squares
A difference of squares is literally and minus sign between two perfect squares. Hence, a difference of squares. Some examples and the factored forms are:

$$4x^2 - 9 = (2x - 3)(2x + 3)$$

$$16x^2 - 25 = (4x - 5)(4x + 5)$$

$$36x^2 - 1 = (6x - 1)(6x + 1)$$

Can you notice any patterns? It is important to make the connection that $4x^2 - 9$, for example, is equivalent to $4x^2 + 0x - 9$. So for all difference of square type polynomials, $b = 0$. This means you could factor by product/sum, decomposition of even algebra tiles but it's easier and more efficient so use logic and patterns.

Conjugates

A **conjugate** is formed by changing the sign between two terms in a binomial. For instance, the **conjugate** of $x + y$ is $x - y$. We can also say that $x + y$ is a **conjugate** of $x - y$. In other words, the two binomials are conjugates of each other. The factors of a difference of squares are conjugates. When factoring a difference of squares, $a^2 - b^2$, you should recognize that the expression is a binomial, the first and last terms are perfect squares and the operation between the terms is subtraction. The factors of the binomial are the square root of the first and last terms where the binomials are conjugates of each other, $(a + b)(a - b)$.

Example 5:

What is the conjugate of:

(A) $(x + 5)$ (B) $(2b - 3)$ (C) $(9 - a)$

$(x-5)$ $(2b+3)$ $(9+a)$

Example 6:

Factor:

(A) $9x^2 - 49$ (B) $81x^2 - 64$ (C) $36 - 121x^2$ (D) $16x^2 - 1$

$(3x-7)(3x+7)$ $(9x-8)(9x+8)$ $(6-11x)(6+11x)$ $(4x-1)(4x+1)$

why?

Expand

$3x(3x+7)-7(3x+7)$

$= 9x^2 + 21x - 21x - 49$

$= 9x^2 \cancel{+0x} - 49$

$= 9x^2 - 49$

Don’t forget if you should factor a GCF out of any polynomial if possible before factoring,

Example 7:
Factor:

(A) $3x^2 - 75$

$= 3(x^2 - 25)$

$= 3(x-5)(x+5)$

(B) $32x^2 - 8$

$= 8(4x^2 - 1)$

$= 8(2x-1)(2x+1)$

(C) $27 - 12x^2$

$= 3(9 - 4x^2)$

$= 3(3-2x)(3+2x)$

Students often, for example, confuse $x^2 + 16$ as a difference of squares. Remember this particular binomial can be written as $x^2 + 0x + 16$. To factor this expression, you need to find two integers with a product of 16 and a sum of zero. If both integers are either positive or negative, a sum of zero is not possible. Therefore the binomial cannot be factored.

Cannot factor a sum of squares!

Example 8:
Explain why $x^2 - 81$ can be factored but $x^2 + 81$ cannot be factored.

$x^2 - 81$ is a difference of squares, factors are $(x-9)(x+9)$

$x^2 + 81$ is a sum of squares. Not factorable!

Example 9:
Find a shortcut for multiplying $(x + 5)(x - 5)$. Explain why this works?

$(x+5)(x-5) = x^2 - 25$. When you multiply a binomial by its conjugate, the linear terms cancel.
(middle)

Example 10:
Explain why $(x - 3)^2 \neq x^2 + 9$.

$(x-3)^2 = (x-3)(x-3)$

$= x(x-3) - 3(x-3)$

$= x^2 - 3x - 3x + 9$

$= x^2 - 6x + 9$

Trinomials With two Variables
The methods used to factor trinomials in one variable are also applicable to trinomials with two variables. You are not expected to model the factoring of trinomials in two variables concretely. For example, you can factor the trinomial $2x^2 - 14xy + 24y^2$ by comparing it to the trinomial in one variable $2x^2 - 14x + 24$. Be careful with the inclusion and placement of variables within the factors.

Example 11:
Factor:

(A) $x^2 - 6xy + 8y^2$

Drop y.

$x^2 - 6x + 8$

$(x-2)(x-4)$

$\frac{8}{1,8 \quad 2,4}$

$\rightarrow (x-2y)(x-4y)$

(B) $6x^2 + xy - 15y^2$

$6x^2 + x - 15$

$(6x^2 + 10x)(-9x - 15)$

$2x(3x+5) - 3(3x+5)$

$(2x-3)(3x+5)$

$\frac{90}{1,90 \quad 2,45 \quad 3,30}$

9,10

$\rightarrow (2x-3y)(3x+5y)$

(C) $4x^2 - 12xy + 9y^2$

$\sqrt{4} = 2$

$\sqrt{9} = 3$

$2(2 \times 3) = 12$ ∴ PST

$(2x-3y)(2x-3y)$

$\rightarrow (2x-3y)^2$

(D) $9x^2 - 25y^2$

$(3x-5y)(3x+5y)$

Textbook Questions: page 194, 195 #4, 5, 6, 8, 10, 11, 12, 13, 14, 18