Math 2200

1.1 Arithmetic Sequences

Sequences
A sequence is an ordered list of objects. It contains elements or terms that follow a pattern or rule to determine the next term in the sequence.

The **first term** of the sequence is t_1. example: $t_3 = 4$

The **number of terms** in the sequence is n. $n = 3$

The **general term** of the sequence is t_n.

Finite sequence
A sequence with a finite number of terms.

Examples: 2, 5, 8, 11, 14 or 5, 10, 15, 20, ..., 100

Infinite Sequence
A sequence with an infinite number of terms. Every term is followed by a new term.

Example: 5, 10, 15, 20, ...

Arithmetic Sequence
An ordered list of terms in which the difference between consecutive terms is constant. Another way to look at this is the same value or variable is added to each term to create the next term.

Common difference: the difference between successive terms in an arithmetic sequence.

Example 1:
Consider a sequence such as 5, 7, 9, 11, 13,...

(A) What is the common difference? 5 7 9 11 13

common difference: 2 $7-5 = 2$ $9-7 = 2$ $11-9 = 2$

(B) Can this sequence be rewritten to show the pattern of the first term and the common difference?

$5, 5+2, 5+2+2, 5+2+2+2, \ldots$

(C) Can you predict the formula for the general term of an arithmetic sequence based on the pattern $5, 5 + (1)2, 5 + 2(2), 5 + 3(2), \ldots$?

$t_n = 5 + 2(n-1)$

$t_n = 5 + 2n - 2$

$t_n = 2n + 3$

(D) Can you write the pattern in general terms for any first term and common difference?

$t_n = t_1 + (n-1) \cdot d$

General term: an expression for directly determining any term of a sequence. The formula is a rule that shows you how the value of t_n depends on n.

The general term of an arithmetic sequence is $t_n = t_1 + (n-1)d$

Example 2:

Consider the sequence: 1, 3, 5, 7,..., 59

(A) Is the sequence arithmetic?

$1, 3, 5, 7 \ldots 59$

$2 \quad 2 \quad 2$

common difference, $d = 2$

$\therefore$ arithmetic

(B) Is the sequence finite or infinite?

(C) Write the general term for the sequence.

$t_1 = 1$

$d = 2$

$t_n = t_1 + (n-1)d$

$t_n = 1 + (n-1) \cdot 2$

$t_n = 1 + 2n - 2$

$t_n = 2n - 1$

Check: $t_4 = 7$

$t_4 = 2(4) - 1$

$t_4 = 8 - 1$

$t_4 = 7$ ✓

(D) What is the 8th term in the sequence?

$t_n = 2n - 1$

$t_8 = 2(8) - 1$

$t_8 = 16 - 1$

$t_8 = 15$

Example 3:
Consider the sequence: 12, 19, 26, 33…

(A) Is the sequence arithmetic?

$$\underbrace{12 \quad 19}_{7} \quad \underbrace{\quad 26}_{7} \quad \underbrace{\quad 33}_{7}$$

$d = 7$

$\therefore$ arithmetic

(B) Is the sequence finite or infinite?

(C) Write the general term for the sequence.

$t_1 = 12$

$d = 7$

$t_n = t_1 + (n-1)d$

$t_n = 12 + (n-1) \cdot 7$

$t_n = 12 + 7n - 7$

$t_n = 7n + 5$

Check:

$t_4 = 33$

$t_n = 7(4) + 5$

$t_n = 28 + 5$

$t_n = 33$

(D) What is the 400th term in the sequence?

$t_n = 7n + 5$

$t_{400} = 7(400) + 5 = 2805$

Example 4:
What is the general term for the sequence: $-40, -43, -46, -49, -52, \ldots$

$t_1 = -40$

$t_n = t_1 + (n-1)d$

$t_n = -40 + (n-1)(-3)$

$t_n = -40 - 3n + 3$

$t_n = -3n - 37$

$-43 - (-40) = -3$

$-46 - (-43) = -3$

$d = -3$

Example 5:

For each arithmetic sequence, write a formula for t_n and use it to find the indicated term.

(A) $-4, 1, 6, 11, \ldots, t_{13}$

$d = 5 \; 5 \; 5$

$t_n = -4 + (n-1) \cdot 5$

$t_n = -4 + 5n - 5$

$t_n = 5n - 9$

$t_{13} = 5(13) - 9$

$t_{13} = 65 - 9$

$t_{13} = 56$

(B) $9, 1, -7, -15, \ldots, t_{46}$

$d = -8 \; -8 \; -8$

$t_n = 9 + (n-1)(-8)$

$t_n = 9 - 8n + 8$

$t_n = -8n + 17$

$t_{46} = -8(46) + 17$

$t_{46} = -351$

Example 6:

Generate the first 6 terms for the arithmetic sequence with general term: $t_n = -3n - 7$

$t_1 = -3(1) - 7 = -3 - 7 = -10$

$t_2 = -3(2) - 7 = -6 - 7 = -13$

$t_3 = -3(3) - 7 = -9 - 7 = -16$

$t_4 = \quad -19$

$t_5 = \quad -22$

$t_6 = \quad -25$

Example 7:

(A) Determine the first five terms for each of the sequences:

i. $t_n = 2n - 1$

$t_1 = 2(1) - 1 = 1$

$t_2 = 2(2) - 1 = 3$

$t_3 = 2(3) - 1 = 5$

$t_4 = 7$

$t_5 = 9$

ii. $t_n = \sqrt{4n^2 - 4n + 1}$

$t_1 = \sqrt{4(1)^2 - 4(1) + 1} = \sqrt{1} = 1$

$t_2 = \sqrt{4(2)^2 - 4(2) + 1} = \sqrt{9} = 3$

$t_3 = \sqrt{4(3)^2 - 4(3) + 1} = \sqrt{25} = 5$

$t_4 = 7$

$t_5 = 9$

iii. $t_n = \frac{2n^2+n-1}{n+1}$

$t_1 = \frac{2(1)^2 + (1) - 1}{1+1} = \frac{2}{2} = 1$

$t_2 = \frac{2(2)^2 + 2 - 1}{2+1} = \frac{9}{3} = 3$

$t_3 = \frac{2(3)^2 + 3 - 1}{3+1} = \frac{20}{4} = 5$

$t_4 = 7$

$t_5 = 9$

(B) Is each sequence [redacted] arithmetic since the first five terms result in {1, 3, 5, 7, 9}? Explain your reasoning.

i. $t_n = 2n - 1$

ii. $t_n = \sqrt{4n^2 - 4n + 1}$

$t_n = \cancel{\sqrt{}}(2n-1)^{\cancel{2}}$

$t_n = 2n - 1$

iii. $t_n = \frac{2n^2 + n - 1}{n+1}$

$t_n = \frac{2n^2 + 2n - n - 1}{n+1}$

$t_n = \frac{2n(n+1) - (n+1)}{n+1}$

$t_n = \frac{(2n-1)\cancel{(n+1)}}{\cancel{n+1}}$

$t_n = 2n - 1$

Yes. All equations simplify to $t_n = 2n - 1$

Example 8:
Explore the following sequences to determine whether each is arithmetic. Justify your conclusions.

(A) 5.3, 5.9, 6.5

0.6 0.6

$\therefore$ arithmetic

(B) $x - 1, x + 1, x + 3$

$x+1-(x-1) = x+1-x+1 = 2$

$x+3-(x+1) = x+3-x-1 = 2$

$\therefore$ arithmetic

(C) x_1, x_2, x_3

$x_2 - x_1$ No way to tell.

$x_3 - x_2$

(D) 1, 2, 4

1 2 $\therefore$ Not arithmetic

(E) $2x + 5, 4x + 5, 6x + 5$

$4x+5-(2x+5) = 4x+5-2x-5 = 2x$

$6x+5-(4x+5) = 6x+5-4x-5 = 2x$

$\therefore$ arithmetic

Example 9:
The first three terms of an arithmetic sequence are: $2x-1, 3x+2, 7x-4$, …
Algebraically determine the value of x, the common difference and all three terms.

$d = 3x+2-(2x-1)$
$d = 3x+2-2x+1$
$d = x+3$
$d = d$
$x+3 = 4x-6$
$3+6 = 4x-x$
$9 = 3x$
$\frac{9}{3} = \frac{3x}{3}$

$\rightarrow x = 3$
$d = 3+3 = 6$

$d = 7x-4-(3x+2)$
$d = 7x-4-3x-2$
$d = 4x-6$

$2(3)-1, 3(3)+2, 7(3)-4$
$5, 11, 7$

Example 10:
Consecutive terms of an arithmetic sequence are $(5+x), 8, (1+2x)$, … Determine the value of x.

$d = 8-(5+x)$
$d = 8-5-x$
$d = 3-x$

$d = 1+2x-8$
$d = 2x-7$

$d = d$
$3-x = 2x-7$
$3+7 = 2x+x$
$10 = 3x$
$\frac{10}{3} = \frac{3x}{3}$
$x = 10/3$

Arithmetic Sequences and Linear Functions

There is a relationship between an arithmetic sequence and a linear function. Let's take a look at this relationship through the following example:

Example 11:

Consider the following data set:

n	1	2	3	4	5
t_n	$t_1 = 3$	5	7	9	11

(A) Find the general term. 2

$2 = d$

$t_n = t_1 + (n-1)d$

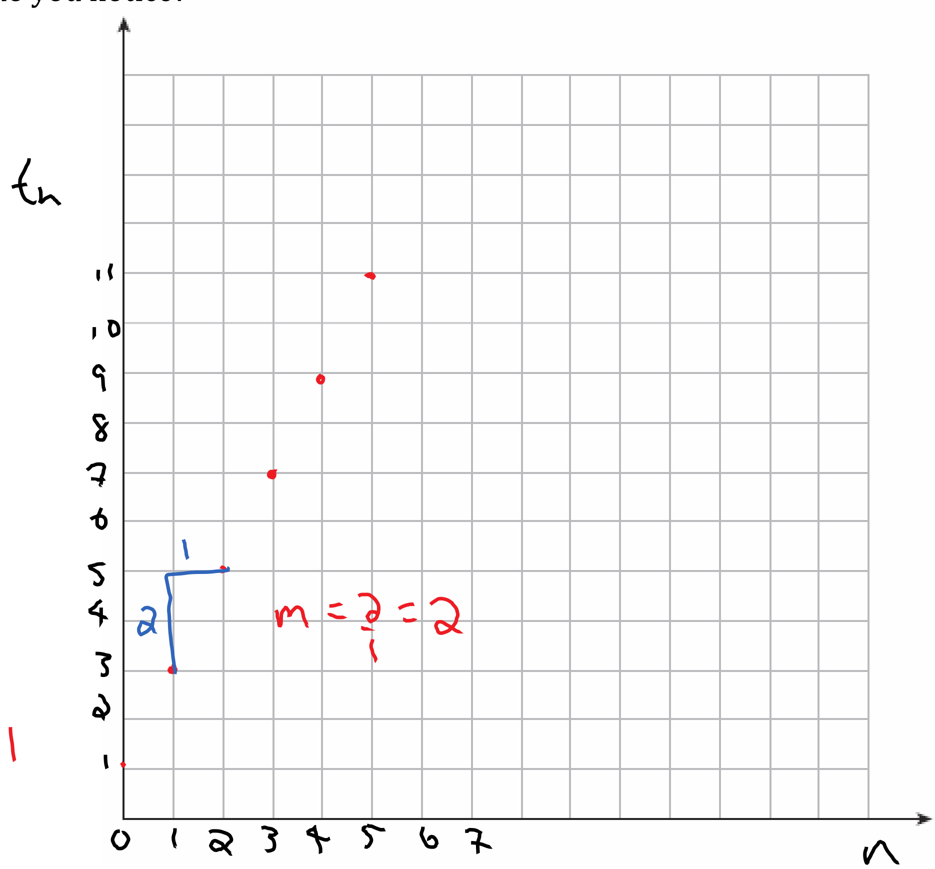
$t_n = 3 + (n-1)2$

$t_n = 3 + 2n - 2$

$t_n = 2n + 1$

(B) Graph the data. What do you notice?

It's linear.



Y-int: 1

(C) Find the equation of the line in $y = mx + b$ form. What is the relationship between the slope of the line and the common difference?

$m = 2$

$b = 1$

$y = 2x + 1$

slope = common difference

(D) Evaluate the first term minus the common difference of the sequence. What does this value represent in the linear equation?

$t_1 - d = 3 - 2 = 1$

The value is the y-intercept.

Example 12:

In an arithmetic sequence, $t_4 = j$ and $t_5 = k$. Find an expression that represents t_8?

$t_4 = j$

$t_5 = k$ $\Big\} d = k - j$

$t_6 = k + k - j = 2k - j$

$t_7 = 2k - j + k - j = 3k - 2j$

$t_8 = 3k - 2j + k - j = 4k - 3j$

Common Mistakes

When provided with the general term, students sometimes have difficulty differentiating between the term and the term number. When they are asked to find the tenth term, for example, they may be unsure whether to write t_{10} or $t_n = 10$. It is important to reinforce that t_{10} represents the tenth term while $t_n = 10$ represents the n^{th} term having a value of 10.

Example 13:
The thirteenth term of an arithmetic sequence is 85 and the twenty-first term is 133. Find the general term of the sequence.

$t_{13} = 85 \qquad t_n = t_1 + (n-1)d$

$t_{21} = 133$

$85 = t_1 + (13-1)d \qquad 133 = t_1 + (21-1)d$

① $85 = t_1 + 12d \qquad$ ② $133 = t_1 + 20d$

$$\begin{array}{rl} & 133 = \cancel{t_1} + 20d \\ - & 85 = \cancel{t_1} + 12d \\ \hline & 48 = 8d \end{array}$$

$\frac{48}{8} = \frac{8d}{8}$

$d = 6$

$85 = t_1 + 12(6)$

$t_1 = 85 - 72$

$t_1 = 13$

$t_n = t_1 + (n-1)d$

$t_n = 13 + (n-1)6$

$t_n = 13 + 6n - 6$

$t_n = 6n + 7$

Example 14:
The musk-ox and the caribou of northern Canada are hoofed mammals that survived the Pleistocene Era, which ended 10 000 years ago. In 1955, the Banks Island musk-ox population was approximately 9250 animals. Suppose that in subsequent years, the growth of its population generated an arithmetic sequence, in which the number of musk-ox increased by approximately 1650 each year. How many years would it take the population to reach 100 000.

$t_1 = 9250$

$d = 1650$

$t_n = 100{,}000$

$n = ?$

$$t_n = t_1 + (n-1)d$$

$$100000 = 9250 + (n-1)1650$$

$$100000 = 9250 + 1650n - 1650$$

$$100000 - 7600 = 1650n$$

$$92400 = 1650n$$

$$\frac{92400}{1650} = \frac{1650n}{1650}$$

$$n = 56$$

It would take 56 years for the population to reach 100,000.

Textbook Questions: page: 16 - 18; # 1, 2, 3, 5, 6, 9, 11, 16