

Math 2200

1.2 Arithmetic Series

Arithmetic Series: a sum of the terms that form an arithmetic sequence. For example:

For the arithmetic sequence, 2, 4, 6, 8

The corresponding arithmetic series would be $2 + 4 + 6 + 8 = 20$

Gauss's Method and the Sum of An Arithmetic Series

Carl Fredrich Gauss was German mathematician born in 1777. Some call him the greatest mathematician that ever lived. A child prodigy, when he was 10 years old, his teacher gave his class a task to keep them busy for what was hoped to be a long time. The teacher was astonished when Gauss gave the correct answer in less than a minute. The task was to add up all the numbers from 1 to 100. The following is what Gauss did:

$$\begin{array}{r} 1 + 2 + 3 + \dots + 98 + 99 + 100 \\ + 100 + 99 + 98 + \dots + 3 + 2 + 1 \\ \hline 101 + 101 + 101 + \dots + 101 + 101 + 101 \end{array}$$

How many 101s are there?

$$100(101) = 10100$$

But there's 2 sets of the series.

$$\frac{10100}{2} = 5050$$

Gauss's method can be used to derive a formula for the sum of the general arithmetic series. The series can be written as:

$$S_n = t_1 + (t_1 + d) + \dots + [t_1 + (n-2)d] + [t_1 + (n-1)d]$$

$$S_n = [t_1 + (n-1)d] + [t_1 + (n-2)d] + \dots + t_1 + d + t_1$$

$$2S_n = [2t_1 + (n-1)d] + [2t_1 + (n-1)d] + \dots + [2t_1 + (n-1)d] + [2t_1 + (n-1)d]$$

$$\frac{2S_n}{2} = \frac{n[2t_1 + (n-1)d]}{2}$$

$$S_n = \frac{n}{2} [2t_1 + (n-1)d]$$

Recall from 1.1: $t_n = t_1 + (n-1)d$

$$S_n = \frac{n}{2} [t_1 + t_1 + (n-1)d]$$

$$S_n = \frac{n}{2} [t_1 + t_n] \leftarrow \text{provided.}$$

The sum of an arithmetic series can be determined using the formula:

$$S_n = \frac{n}{2}[2t_1 + (n - 1)d]$$

Where t_1 is the first term

n is the number of terms

d is the common difference

S_n is the sum of the first n terms

However, we can substitute the formula for the general term, t_n as follows:

$$S_n = \frac{n}{2}[2t_1 + (n - 1)d]$$

$$S_n = \frac{n}{2}[t_1 + t_1 + (n - 1)d]$$

$$S_n = \frac{n}{2}[t_1 + t_n]$$

Where t_1 is the first term

n is the number of terms

t_n is the n^{th} term

S_n is the sum of the first n terms

Example 1:

(A) Use Gauss's method to determine the sum of the first 12 positive integers.

$$\begin{array}{r} 1 + 2 + 3 + \dots + 10 + 11 + 12 \\ + 12 + 11 + 10 + \dots + 3 + 2 + 1 \\ \hline 13 + 13 + 13 + \dots + 13 + 13 + 13 \end{array}$$

$$12(13) = \frac{156}{2} = 78$$

(B) Use the formula for an arithmetic series to find the sum of the first 12 positive integers.

$$S_n = \frac{n}{2} [t_1 + t_n]$$

$$n = 12$$

$$t_1 = 1$$

$$t_{12} = 12$$

$$S_{12} = \frac{12}{2} [1 + 12]$$

$$S_{12} = 6 [13]$$

$$S_{12} = 78$$

Example 2:

Find the sum of $20 + 14 + 8 + \dots + (-70)$

$$S_n = \frac{n}{2} [t_1 + t_n]$$

$$t_1 = 20$$

$$t_n = -70$$

$$t_n = t_1 + (n-1)d$$

$$-70 = 20 + (n-1)(-6)$$

$$-70 = 20 - 6n + 6$$

$$-70 = 26 - 6n$$

$$-70 - 26 = -6n$$

$$-96 = -6n$$

$$n = \frac{-96}{-6} = 16$$

$$20, 14, 8, \dots, (-70)$$

$$d = \begin{matrix} \sqrt{} & \sqrt{} \\ -6 & -6 \end{matrix}$$

$$S_{16} = \frac{16}{2} [20 + (-70)]$$

$$S_{16} = 8[-50]$$

$$S_{16} = -400$$

Final Exam!

Example 3:

An auditorium has 20 seats on the first row, 24 seats on the second row, 28 seats on the third row, and so on, and has 30 rows of seats. How many seats are in the theatre?

$$20 + 24 + 28 + \dots + t_{30} \quad n = 30$$

$$d = \begin{array}{c} \vee \\ 4 \end{array} \quad \begin{array}{c} \vee \\ 4 \end{array}$$

$$t_{30} = 20 + (30-1)(4) \quad \text{or} \quad S_n = \frac{n}{2} [t_1 + t_1 + (n-1)d]$$

$$t_{30} = 20 + 29(4)$$

$$t_{30} = 20 + 116$$

$$t_{30} = 136$$

$$S_{30} = \frac{30}{2} [20 + 136]$$
$$= 15(156) = 2340$$

Replace t_n with $t_1 + (n-1)d$

$$S_n = \frac{n}{2} [2t_1 + (n-1)d]$$

$$S_{30} = \frac{30}{2} [2(20) + (30-1)(4)]$$

$$S_{30} = 15 [40 + 116]$$

$$S_{30} = 15(156) = 2340$$

Example 4:

Show that $1 + 3 + 5 + \dots + (2n-1) = n^2$

$$S_n = \frac{n}{2} [t_1 + t_n]$$

$$t_1 = 1$$

$$t_n = 2n-1$$

$$S_n = \frac{n}{2} [1 + (2n-1)]$$

$$S_n = \frac{n}{2} [2n]$$

$$S_n = n(n)$$

$$S_n = n^2$$

Example 5:

In an arithmetic series $t_1 = 6$ and $S_9 = 108$. Ask students to find the common difference and the sum of the first 21 terms.

$$S_n = \frac{n}{2} [t_1 + t_n]$$

$$108 = \frac{9}{2} [6 + 6 + 8d]$$

$$108 = 4.5 [12 + 8d]$$

$$108 = 54 + 36d$$

$$108 - 54 = 36d$$

$$\frac{54}{36} = \frac{36d}{36}$$

$$d = \frac{3}{2} = 1.5$$

$$t_n = t_1 + (n-1)d$$

$$t_9 = 6 + (9-1)d$$

$$t_9 = 6 + 8d$$

$$t_{21} = 6 + (21-1)(1.5)$$

$$t_{21} = 6 + 30$$

$$t_{21} = 36$$

$$S_{21} = \frac{21}{2} [6 + 36]$$

$$S_{21} = 10.5 [42]$$

$$S_{21} = 441$$

Example 6:

The sum of the first two terms of an arithmetic series is 13 and the sum of the first four terms is 46. Determine the first six terms of the series and the sum of the series to six terms.

$$S_2 = 13, S_4 = 46 \quad S_n = \frac{n}{2} [2t_1 + (n-1)d]$$

$$13 = \frac{2}{2} [2t_1 + (2-1)d] \quad 46 = \frac{4}{2} [2t_1 + (4-1)d]$$

$$\textcircled{1} 13 = 2t_1 + d$$

$$46 = 2[2t_1 + 3d]$$

$$\frac{46}{2} = \frac{4t_1 + 6d}{2}$$

$$\textcircled{1} 23 = 2t_1 + 3d$$

$$\textcircled{1} 23 = 2t_1 + 3d$$

$$-\textcircled{2} 13 = 2t_1 + d$$

$$\frac{10}{2} = \frac{2d}{2}$$

$$d = 5$$

Sub d into $\textcircled{2}$

$$13 = 2t_1 + 5$$

$$13 - 5 = 2t_1$$

$$8 = 2t_1$$

$$\frac{8}{2} = \frac{2t_1}{2}$$

$$t_1 = 4$$

$$4 + 9 + 14 + 19 + 24 + 29$$

$$S_6 = \frac{6}{2} (4 + 29) = 3(33) = 99$$