Math 2200

1.3 Geometric Sequences

Geometric Sequence: a sequence in which the ratio of consecutive terms is constant.

Common ratio: the ratio of successive terms in a geometric sequence. The ratio may be positive or negative.

$n = 4$

$$r = \frac{t_n}{t_{n-1}} \qquad r = \frac{t_4}{t_{4-1}} = \frac{t_4}{t_3}$$

For example, for the following sequence, 2, 4, 8, 16, … , the ratio is:

$\frac{t_3}{t_2} = \frac{4}{2} = 2$, $\frac{t_4}{t_3} = \frac{16}{8} = 2$ $\quad r = 2 \quad \therefore$ geometric

Example 1:
Determine if each sequence is geometric:

(A) 3, 9, 27, …

$\frac{t_2}{t_1} = \frac{9}{3} = 3$ $\quad r = 3 \quad \therefore$ geometric

$\frac{t_3}{t_2} = \frac{27}{9} = 3$

(B) 2, 6, 10, 14, …

$\frac{t_2}{t_1} = \frac{6}{2} = 3$

No common ratio $\therefore$ not geometric

$\frac{t_3}{t_2} = \frac{10}{6} = 5/3$

To develop the formula for a geometric sequence, consider a sequence such as 1, 3, 9, 27, 81, … .

i. What is the common ratio?

$$\frac{t_2}{t_1} = \frac{3}{1} = 3$$

common ratio, $r = 3$

$$\frac{t_5}{t_4} = \frac{81}{27} = 3$$

ii. Can this sequence be rewritten to show the pattern of the first term and the common ratio

$$1, \quad 3, \quad 9, \quad 27, \quad 81, \ldots$$

$$1 \cdot 1, \quad 1 \cdot 3, \quad 1 \cdot 3 \cdot 3, \quad 1 \cdot 3 \cdot 3 \cdot 3, \quad 1 \cdot 3 \cdot 3 \cdot 3 \cdot 3, \ldots$$

$$1 \cdot 3^0, \quad 1 \cdot 3^1, \quad 1 \cdot 3^2, \quad 1 \cdot 3^3, \quad 1 \cdot 3^4, \ldots$$

iii. Can you write the pattern in general terms for any first term and common ratio ?

$$t_n = t_1 \cdot r^{n-1}$$

The general term of a geometric sequence where n is a positive integer is

$$t_n = t_1 r^{n-1}$$ ← provided

where t_1 is the first term of the sequence
n is the number of terms
r is the common ratio
t_n is the general or n^{th} term

Example 2:
Generate the first five terms of the sequence: $t_n = 2(3)^{n-1}$.

$t_1 = 2(3)^{1-1} = 2(3)^0 = 2 \cdot 1 = 2$
$t_2 = 2(3)^{2-1} = 2(3)^1 = 2 \cdot 3 = 6$ or $2, 6, 18, 54, 162, \ldots$
$t_3 = 2(3)^{3-1} = 2(3)^2 = 2 \cdot 9 = 18$
$\vdots$

Example 3:
Determine the formula for the n^{th} term: $t_n = t_1 \cdot r^{n-1}$

(A) $184, -92, 46, -23, \ldots$

$t_1 = 184$
$r = ?$

$\frac{t_2}{t_1} = \frac{-92}{184} = -\frac{1}{2}$

$\frac{t_3}{t_2} = \frac{46}{-92} = -\frac{1}{2}$

$\therefore \; t_n = 184 \cdot \left(-\frac{1}{2}\right)^{n-1}$

*Note: You can have negative ratios.

(B) 7, 21, 63, 189, ...

$$\frac{t_2}{t_1} = \frac{21}{7} = 3$$

$$\frac{t_3}{t_2} = \frac{63}{21} = 3$$

$$t_1 = 7$$

$$\therefore \boxed{t_n = 7 \cdot 3^{n-1}}$$

$$\cancel{t_n = 21^{n-1}}$$ ⟵ common mistake

- Recall rules of exponents

Example 4:
The first three terms of a geometric sequence are $\{x - 1, 2x, 3x + 9, \dots\}$. Algebraically determine the value of x.

$$\frac{t_2}{t_1} = \frac{2x}{x-1}, \quad \frac{t_3}{t_2} = \frac{3x+9}{2x}$$

$$\frac{2x}{x-1} = \frac{3x+9}{2x}$$

$$(2x)(2x) = (3x+9)(x-1)$$

$$4x^2 = 3x^2 + 6x - 9$$

$$4x^2 - 3x^2 - 6x + 9 = 0$$

$$x^2 - 6x + 9 = 0$$

$$(x-3)(x-3) = 0$$

$$x = 3$$

Common Exponents
If an exponential equation has the same exponent, the bases must be equal as well. We can use this fact to help us solve equations involving geometric sequences. We can also use the properties of n^{th} roots.

ie if $n^4 = 3^4$, then $n = 3$

Example 5:
Determine the formula for the n^{th} term: $t_3 = 5, t_6 = 135$

Algebraically:

$t_n = t_1 \cdot r^{n-1}$

$5 = t_1 \cdot r^{3-1}$, $135 = t_1 r^{6-1}$

① $5 = t_1 \cdot r^2$ ② $135 = t_1 r^5$

Solve ① for t_1 → Sub ① into ②

$\frac{5}{r^2} = \frac{t_1 \cancel{r^2}}{\cancel{r^2}}$

$t_1 = \frac{5}{r^2}$

$135 = \frac{5}{\cancel{r^2}} \cdot r^{3}$ or $\frac{r^5}{r^2} = r^{5-2} = r^3$

$\frac{135}{5} = \frac{5r^3}{5}$

$r^3 = 27$

$r^3 = 3^3$

$r = 3$

or

Sub r into ①

$t_1 = \frac{5}{3^2} = \frac{5}{9}$

$\therefore t_n = \frac{5}{9} \cdot 3^{n-1}$

Logically:

$t_3 = 5$, $t_6 = 135$

$\frac{135}{5} = 27$

$r^3 = 27$

$\sqrt[3]{r^3} = \sqrt[3]{27}$

$r = 3$

$t_2 = t_1 \cdot r$

$t_3 = t_1 \cdot r \cdot r$

$\frac{t_3}{r^2} = \frac{t_1 r^2}{r^2}$

$t_1 = \frac{t_3}{r^2} = \frac{5}{3^2} = \frac{5}{9}$

$\therefore t_n = \frac{5}{9} \cdot 3^{n-1}$

Example 6:
In a geometric sequence, the third term is 54 and the sixth term is -1458. Determine the values of t_1 and r, and list the first three terms of the sequence.

$t_3 = 54$, $t_6 = -1458$

$\frac{-1458}{54} = -27$

$r^3 = -27$

$\sqrt[3]{r^3} = \sqrt[3]{-27}$

$r = -3$

$t_2 = \frac{54}{-3} = -18$

$t_1 = \frac{-18}{-3} = 6$

$\therefore t_n = 6 \cdot (-3)^{n-1}$

$6, -18, 54, -162, \ldots$ alternating sequence

Example 7:
In a geometric sequence, the second term is 28 and the fifth term is 1792. Determine the values of t_1 and r, and list the first three terms of the sequence.

$t_2 = 28$, $t_5 = 1792$

$t_1 = \frac{28}{4} = 7$

$r^3 = \frac{1792}{28}$

$t_n = 7 \cdot 4^{n-1}$

$r^3 = 64$

7, 28, 112, ...

$\sqrt[3]{r^3} = \sqrt[3]{64}$

$r = 4$

Textbook Questions: page: 39 - 44; # 1, 3, 5, 6, 7, 10, 14, 16, 23, 25