Math 2200

1.4 Geometric Series

Geometric Series: the expression for the sum of the terms of a geometric sequence.

For example, if

$3, 6, 12, 24, \ldots$ is a geometric sequence

$3 + 6 + 12 + 24 + \cdots$ is the corresponding geometric series.

The sum of a geometric series can be determined using the formula:

$$S_n = \frac{t_1(r^n - 1)}{r - 1}, r \neq 1$$

provided

Where t_1 is the first term of the series
- n is the number of terms
- r is the common ratio
- S_n is the sum of the first n terms

Example 1:
Determine the sum of the first 10 terms of each geometric sequence:

(A) $4 + 12 + 36 + \cdots$

$t_1 = 4$

$r = \frac{12}{4} = 3$

$n = 10$

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{4(3^{10} - 1)}{3 - 1}$$

$$S_{10} = \frac{\cancel{4}^{2}(59048)}{\cancel{2}}$$

$$S_{10} = 2(59048)$$

$$S_{10} = 118\,096$$

(B) $t_1 = 5, r = \frac{1}{2}$, $n=10$

$$S_{10} = \frac{5\left[\left(\frac{1}{2}\right)^{10} - 1\right]}{\frac{1}{2} - 1}$$

$$S_{10} = \frac{5\left(\frac{1}{1024} - \frac{1024}{1024}\right)}{-\frac{1}{2}}$$

$$S_{10} = 5\left(\frac{-1023}{1024}\right)\cdot\left(-\frac{2}{1}\right)$$

$$\rightarrow S_{10} = -10\left(-\frac{1023}{1024}\right)$$

$$S_{10} = \frac{10230}{1024}$$

$$S_{10} = \frac{5115}{512}$$

Common Bases
We can also use the concept of common bases to find the value of n. It makes sense to think that if two exponential expressions are equal and have the same base, the exponents must be equal.

Example 2:
Solve for n:

(A) $324 = 4(3)^n$ * Isolate the base and exponent.

$$\frac{324}{4} = \frac{4(3)^n}{4}$$

$$81 = 3^n$$

$$3^4 = 3^n$$

$$n = 4$$

(B) $\frac{1}{32} = \left(\frac{1}{2}\right)^{n-1}$

$\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^{n-1}$

$5 = n-1$

$5+1 = n$

$n = 6$

(C) $192 = 3(2)^{n-1}$

$\frac{192}{3} = \frac{3(2)^{n-1}}{3}$

$64 = 2^{n-1}$

$2^6 = 2^{n-1}$

$\rightarrow 6 = n-1$

$6+1 = n$

$n = 7$

Example 3:
Determine the sum of each geometric series.

$S_n = \frac{t_1(r^n - 1)}{r-1}$

(A) $4 - 16 + 64 - \cdots - 65536$

$t_1 = 4$

$r = \frac{-16}{4} = -4$

$n = ?$

- use $t_n = t_1 r^{n-1}$ to find n.

$t_n = -65536$

$\frac{-65536}{4} = \frac{4(-4)^{n-1}}{4}$

$-16384 = (-4)^{n-1}$

$(-4)^7 = (-4)^{n-1}$

$7 = n-1$

$n = 8$

$S_8 = \frac{4[(-4)^8 - 1]}{-4-1}$

$S_8 = \frac{4(65535)}{-5}$

$S_8 = 4(-13107)$

$S_8 = -52428$

(B) $\frac{1}{27}+\frac{1}{9}+\frac{1}{3}+\cdots+729$

$t_1=\frac{1}{27}$

$r=\frac{\frac{1}{9}}{\frac{1}{27}}=\frac{1}{9}\cdot\frac{27}{1}=3$

$n=?$

$t_n=t_1\cdot r^{n-1}$

$27\cdot 729=\frac{1}{\cancel{27}}\cdot 3^{n-1}\cdot\cancel{27}$

$19683=3^{n-1}$

$3^9=3^{n-1}$

$9=n-1$

$n=10$

$S_n=\frac{t_1(r^n-1)}{r-1}$

$S_{10}=\frac{\frac{1}{27}(3^{10}-1)}{3-1}$

$S_{10}=\frac{\frac{1}{27}(59048)}{2}$

$S_{10}=\frac{59048}{27}\cdot\frac{1}{2}$

$S_{10}=\frac{29524}{27}$

Your turn…

Example 4:
Determine the sum of the following geometric series.

(A) $\frac{1}{64}+\frac{1}{16}+\frac{1}{4}+\cdots+1024$

$t_1=\frac{1}{64}$

$r=\frac{\frac{1}{16}}{\frac{1}{64}}=\frac{1}{16}\cdot\frac{64}{1}=4$

$t_n=1024$

$64\cdot 1024=\frac{1}{\cancel{64}}(4)^{n-1}\cdot\cancel{64}$

$65536=4^{n-1}$

$4^8=4^{n-1}$

$8=n-1$

$n=8+1$

$n=9$

$S_9=\frac{\frac{1}{64}(4^9-1)}{4-1}$

$S_9=\frac{\frac{1}{64}(262143)}{3}$

$S_9=\frac{262143}{192}$

$S_9=\frac{87381}{64}$

(B) $-2+4-8+\cdots-8192$

$$t_1=-2$$

$$r=\frac{4}{-2}=-2$$

$$t_n=-8192$$

$$\frac{-8192}{-2}=\frac{\cancel{-2}(-2)^{n-1}}{\cancel{-2}}$$

$$4096=(-2)^{n-1}$$

$$(-2)^{12}=(-2)^{n-1}$$

$$12=n-1$$

$$12+1=n$$

$$n=13$$

$$S_{13}=\frac{-2\left[(-2)^{13}-1\right]}{-2-1}$$

$$S_{13}=\frac{-2(-8193)}{-3}$$

$$S_{13}=\frac{16386}{-3}$$

$$S_{13}=-5462$$

Example 5:
Algebraically determine the number of terms in the geometric series,

$\frac{1}{81} + \frac{1}{27} + \frac{1}{9} + \cdots + 2187$, and find the sum of the series.

$t_1 = \frac{1}{81}$

$r = \frac{\frac{1}{27}}{\frac{1}{81}} = \frac{1}{27} \cdot \frac{81}{1} = 3$

$t_n = 2187$

$81 \cdot 2187 = \frac{1}{\cancel{81}}(3)^{n-1} \cdot \cancel{81}$

$177147 = 3^{n-1}$

$3^{11} = 3^{n-1}$

$11 = n - 1$

$11 + 1 = n$

$n = 12$

$S_{12} = \frac{\frac{1}{81}(3^{12} - 1)}{3 - 1}$

$S_{12} = \frac{\frac{1}{81}(531440)}{2}$

$S_{12} = \frac{531440}{162}$

$S_{12} = \frac{265720}{81}$

Steps for Word Problems
When working through the problems, remember the following guidelines:

- Draw a diagram if necessary.
- Write out the terms of the sequence.
- Determine if the problem is a sequence or series.
- Determine if the problem is arithmetic or geometric.
- Construct the formula using the given information in the problem.
- Solve the problem.

Example 6:

A student is constructing a family tree. He is hoping to trace back through 10 generations to calculate the total number of ancestors he has. Determine the total number of ancestors after the 10^{th} generation.

$2, 4, 8, \ldots$

$t_1 = 2$

$r = \frac{4}{2} = 2$

$r = \frac{8}{4} = 2$

$n = 10$

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{2(2^{10} - 1)}{2 - 1}$$

$$S_{10} = \frac{2(1023)}{1}$$

$$S_{10} = 2046$$

Example 7:
Consider a ball that is dropped from a height of 38.28 m and bounces back up 60% of the original height. Find the total distance travelled by the ball by the time it hits the ground for the tenth time.

$38.28, \quad 38.28 \cdot 0.6 = 22.97, \quad 22.97 \cdot 0.6 = 13.78, \ldots$

$S_9 = 22.97 + 13.78 + \ldots \times 2$

$$S_9 = \frac{22.97(0.6^9 - 1)}{0.6 - 1}$$

$$S_9 = \frac{-22.739}{-0.4}$$

$S_9 = 56.84$

Total distance: $38.28\text{m} + 2(56.84\text{m})$
$= 152.0\text{m}$

Textbook Questions: page: 53 - 56; # 1, 2, 3, 4, 5, 6, 7, 8, 10, 25