Math 2200

1.5 Infinite Geometric Series

Convergent Series: a series with an infinite number of terms, in which the sequence of **partial sums** approaches a fixed value.

For example, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

Divergent Series: a series with an infinite number of terms, in which the sequence of **partial sums** does not approach a fixed value.

For example, $2 + 4 + 8 + 16 + \cdots$

A **Partial Sum** is the **sum** of part of the sequence. The **partial sum** of the first 4 terms of that sequence: $2, 4, 6, 8, 10, 12, \ldots$ is $2 + 4 + 6 + 8 = 20$.

Let's take a look at some examples of **convergent** and **divergent** series.

Geometric Series	Partial Sum	Convergent/ Divergent
$2 + 4 + 8 + 16 + \ldots$ $(r > 1)$	$S_1 = 2$, $S_2 = 6$, $S_3 = 14$	diverges
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots.$ $(-1 < r < 1)$	$S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$	converges to 1
$-1 + 1 + -1 + 1 + \ldots.$ $(r = -1)$	$S_1 = -1$, $S_2 = 0$, $S_3 = -1$ $S_4 = 0$	diverges
$1 + 1 + 1 + 1 + \ldots$ $(r = 1)$	$S_1 = 1$, $S_2 = 2$, $S_3 = 3$, $S_4 = 4$	diverges

What do you notice about the ratio of the only sequence that converges?

r is between -1 and 1.

The sum of an infinite geometric series, where $-1 < r < 1$, can be determined using the formula

$$S_\infty = \frac{t_1}{1-r}$$

Where t_1 is the first term of the series
r is the common ratio
S_∞ represents the sum of an infinite number of terms

Example 1:
Determine whether the following sequences converge or diverge:

(A) $8, 4, 2, 1, 0.5, \ldots$

$r = \frac{4}{8} = \frac{1}{2}$, $r = \frac{2}{4} = \frac{1}{2}$ $\quad -1 < \frac{1}{2} < 1$

$\therefore$ converges

(B) $3, \frac{7}{3}, \frac{5}{3}, 1, \frac{1}{3}, \ldots$

$r = \frac{\frac{7}{3}}{\frac{3}{1}} = \frac{7}{3} \cdot \frac{1}{3} = \frac{7}{9}$, $r = \frac{\frac{5}{3}}{\frac{7}{3}} = \frac{5}{3} \cdot \frac{3}{7} = \frac{5}{7}$

$\therefore$ not geometric

(C) $5^{-3}, 5^{-2}, 5^{-1}, 5^{0}, \ldots$

$r = \frac{5^{-2}}{5^{-3}} = 5^{-2+3} = 5$, $r = \frac{5^{-1}}{5^{-2}} = 5^{-1+2} = 5$

$\therefore$ divergent

(D) $t_1 + d, t_1 + 2d, t_1 + 3d, t_1 + 4d, \ldots$

$r = \frac{t_1 + 2d}{t_1 + d}$, $r = \frac{t_1 + 3d}{t_1 + 2d}$

$\therefore$ not geometric

Example 2:
Decide whether each infinite geometric series is convergent or divergent. Find the sum of the series if it exists.

(A) $4 + 2 + 1 + 0.5 + 0.25 + \cdots$

$r = \frac{2}{4} = \frac{1}{2}$, $r = \frac{1}{2}$

$-1 < \frac{1}{2} < 1$

$\therefore$ convergent

$$S_\infty = \frac{t_1}{1-r} = \frac{4}{1-\frac{1}{2}} = \frac{4}{\frac{1}{2}} = 4 \cdot 2 = 8$$

(B) $2 - 4 + 8 - \cdots$

$r = \frac{-4}{2} = -2$, $r = \frac{8}{-4} = -2$

$-2 < -1$ $\therefore$ divergent

(C) $1 - \frac{1}{3} + \frac{1}{9} - \cdots$

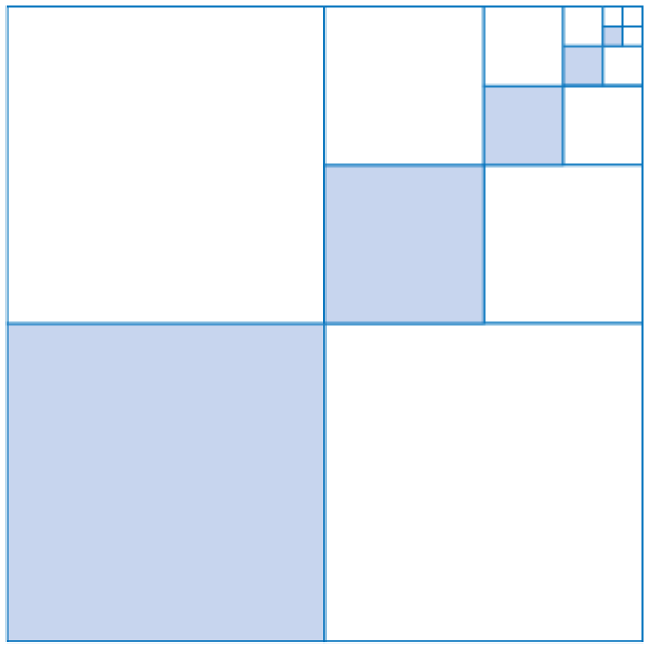
$r = \frac{-\frac{1}{3}}{1} = -\frac{1}{3}$, $r = \frac{\frac{1}{9}}{-\frac{1}{3}} = \frac{1}{9} \cdot -\frac{3}{1} = -\frac{1}{3}$

$-1 < -\frac{1}{3} < 1$

$$S_\infty = \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

Example 3:

Assume that each shaded square represents $\frac{1}{4}$ of the area of the larger square bordering two of its adjacent sides and the shading continues indefinitely in the indicated manner.



(A) Write the series of terms that would represent this situation.

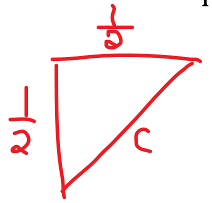
$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$$

(B) How much of the total area of the largest square is shaded.

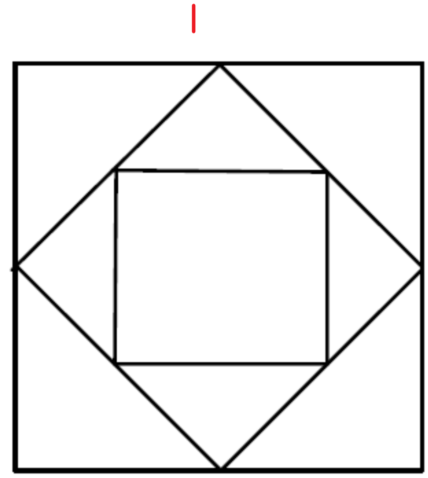
$$r = \frac{\frac{1}{16}}{\frac{1}{4}} = \frac{1}{16} \cdot \frac{4}{1} = \frac{1}{4} \qquad -1 < \frac{1}{4} < 1$$

$$S_\infty = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{\cancel{4}} \cdot \frac{\cancel{4}}{3} = \frac{1}{3}$$

Example 4:
The midpoints of a square with sides 1 m long are joined to form another square. Then the midpoints of the sides of the second square are joined to form a third square. This process is continued indefinitely to form an infinite set of smaller and smaller squares converging on the center of the original square. Determine the total length of the segments forming the sides of all the squares.



$$\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 = c^2$$
$$\tfrac{1}{4} + \tfrac{1}{4} = c^2$$
$$\sqrt{\tfrac{1}{2}} = \sqrt{c^2}$$
$$c = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$
$$c = \frac{\sqrt{2}}{2}$$



$$4(1) + 4\left(\frac{\sqrt{2}}{2}\right)$$
$$4 + 2\sqrt{2} + \dots$$
$$r = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$$

$$S_\infty = \frac{4}{1 - \frac{\sqrt{2}}{2}}$$
$$S_\infty = \frac{4}{\frac{2-\sqrt{2}}{2}}$$
$$S_\infty = 4 \cdot \frac{2}{2-\sqrt{2}}$$
$$S_\infty = \frac{8}{2-\sqrt{2}} \cdot \frac{(2+\sqrt{2})}{(2+\sqrt{2})}$$
$$= \frac{16+8\sqrt{2}}{4-2}$$
$$= \frac{16+8\sqrt{2}}{2}$$
$$= 8+4\sqrt{2}$$

Textbook Questions: page: 63 - 65; # 1, 2, 3, 5, 6, 7, 8, 9, 12

Congratulations on completing all required material for Advanced Mathematics 2200!