Math 2200 **Date:**____________________

2.2A Trigonometric Ratios of Any Angle

In Mathematics 1201, you applied the primary trigonometric ratios to angles between 0° and 90°. We will now explore angles between 0° and 360° using coordinates and reference angles.

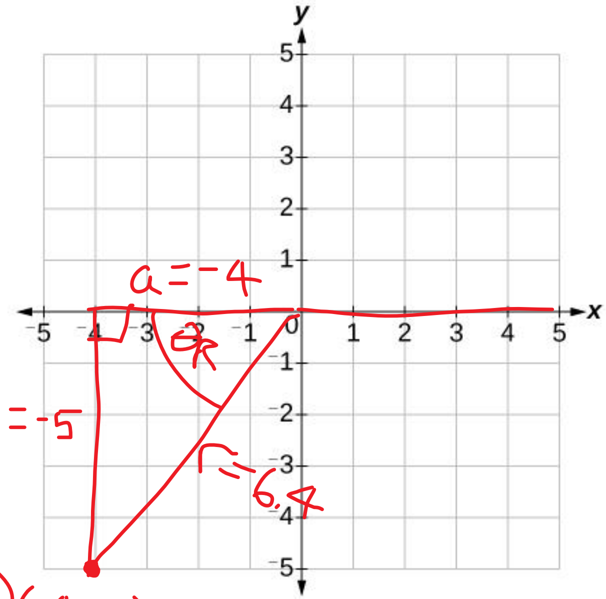
Example 1:

(A) Plot the given point $P(-4, -5)$.
What quadrant does this point lie in?

QIII

(B) Construct the corresponding angle in standard position.

(C) Drop a perpendicular to the x-axis creating a right triangle. Which value represents the adjacent side? Which value represents the opposite side?



(D) How can you determine the exact length of the hypotenuse? Pythagorean Theorem

$(-4)^2 + (-5)^2 = r^2$ $\sqrt{r^2} = \sqrt{41}$

$16 + 25 = r^2$ $r = 6.4$

$41 = r^2$

(E) State the cosine, sine and tangent ratios associated with the angle.

$\cos\theta = -\frac{4}{6.4}$ $\sin\theta = -\frac{5}{6.4}$ $\tan\theta = \frac{-5}{-4}$

$\cos\theta = -0.6250$ $\sin\theta = -0.7813$ $\tan\theta = 1.2500$

(F) What determines the sign of the ratio? Explain your reasoning.

The adjacent and opposite sides determine the sign of the ratio. The terminal arm, r, will always be positive.

In Section 2.1, we learned the relationship between an **angle in standard position** and its corresponding **reference angle**. We then looked at the trigonometric ratios of special angles in the first quadrant. In this section we will explore trigonometric ratios of any angle and how they relate to a reference angle and the angle in standard position.

Finding Trigonometric Ratios of Any Angle, θ

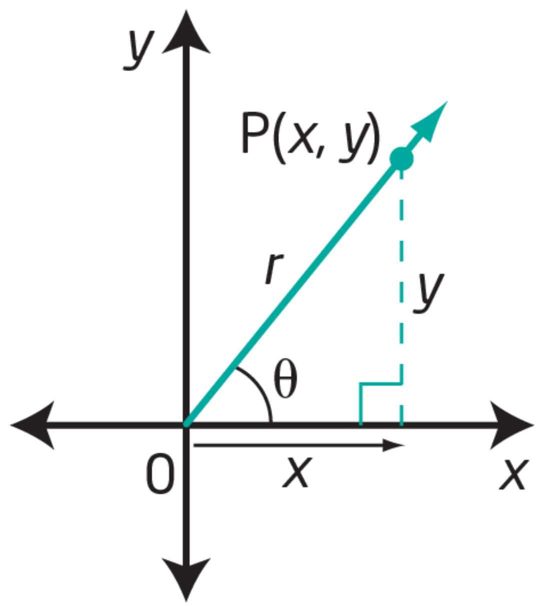
Suppose θ is any angle in standard position, and P(x, y) is any point on it's terminal arm at a distance r from the origin.

Then by Pythagorean Theorem:

$$r^2 = x^2 + y^2$$

$$\sqrt{r^2} = \sqrt{x^2 + y^2}$$

$$r = \sqrt{x^2 + y^2}$$



You can then use a reference angle to determine the three primary trigonometric ratios in terms of x, y, and r.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

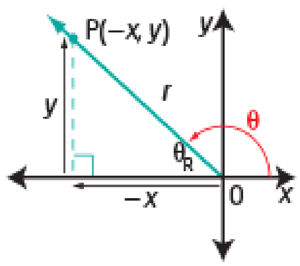
$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

The following chart summarizes the signs of the trigonometric ratios in each quadrant.

Quadrant II
$90° < \theta < 180°$

$\sin\theta = \frac{y}{r}$ $\quad$ $\cos\theta = \frac{-x}{r}$ $\quad$ $\tan\theta = \frac{y}{-x}$

$\sin\theta > 0$ $\quad$ $\cos\theta < 0$ $\quad$ $\tan\theta < 0$

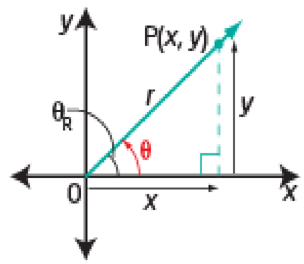


$\theta = 180° - \theta_R$

Quadrant I
$0° < \theta < 90°$

$\sin\theta = \frac{y}{r}$ $\quad$ $\cos\theta = \frac{x}{r}$ $\quad$ $\tan\theta = \frac{y}{x}$

$\sin\theta > 0$ $\quad$ $\cos\theta > 0$ $\quad$ $\tan\theta > 0$



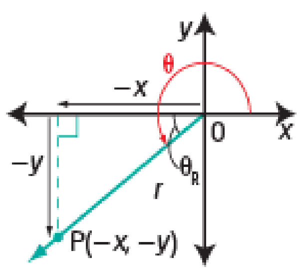
Why is r always positive?

$\theta = \theta_R$

Quadrant III
$180° < \theta < 270°$

$\sin\theta = \frac{-y}{r}$ $\quad$ $\cos\theta = \frac{-x}{r}$ $\quad$ $\tan\theta = \frac{-y}{-x}$

$\sin\theta < 0$ $\quad$ $\cos\theta < 0$ $\quad$ $\tan\theta > 0$

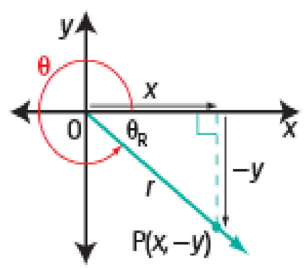


$\theta = 180° + \theta_R$

Quadrant IV
$270° < \theta < 360°$

$\sin\theta = \frac{-y}{r}$ $\quad$ $\cos\theta = \frac{x}{r}$ $\quad$ $\tan\theta = \frac{-y}{x}$

$\sin\theta < 0$ $\quad$ $\cos\theta > 0$ $\quad$ $\tan\theta < 0$



$\theta = 360° - \theta_R$

Example 2:
The point P(−3, 4) lies on the terminal arm of an angle, θ, in standard position. Determine the exact trigonometric ratio for cosθ, sinθ and tanθ.

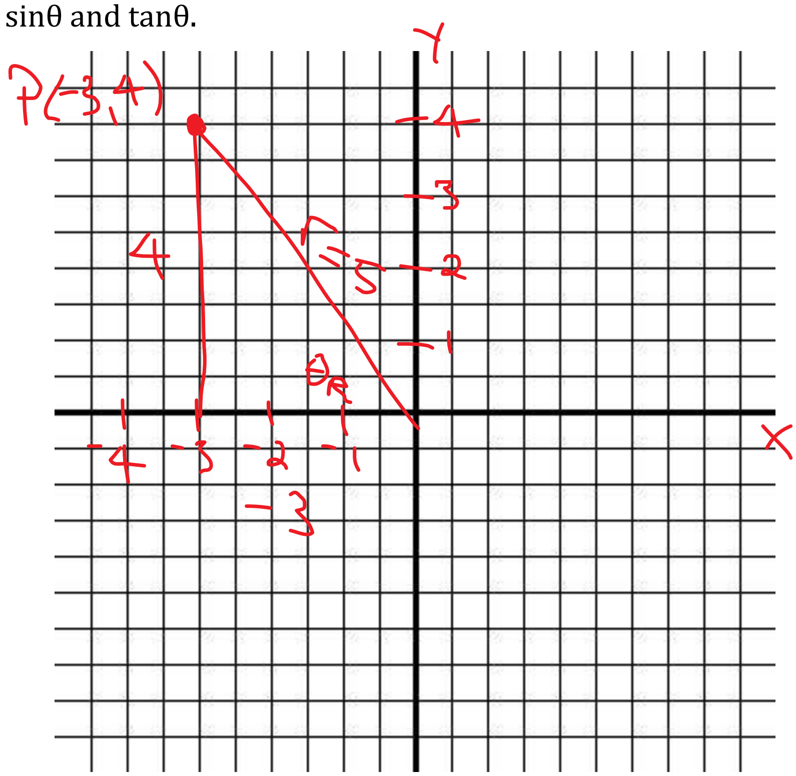
Pythagorean Triples

3-4-5 triangle

$\cos\theta = -\frac{3}{5}$

$\sin\theta = \frac{4}{5}$

$\tan\theta = \frac{4}{-3} = -\frac{4}{3}$



Example 3:
The point P(−8, −15) lies on the terminal arm of an angle, θ, in standard position. Determine the exact trigonometric ratio for cosθ, sinθ and tanθ.

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{(-8)^2 + (-15)^2}$

$r = \sqrt{64 + 225}$

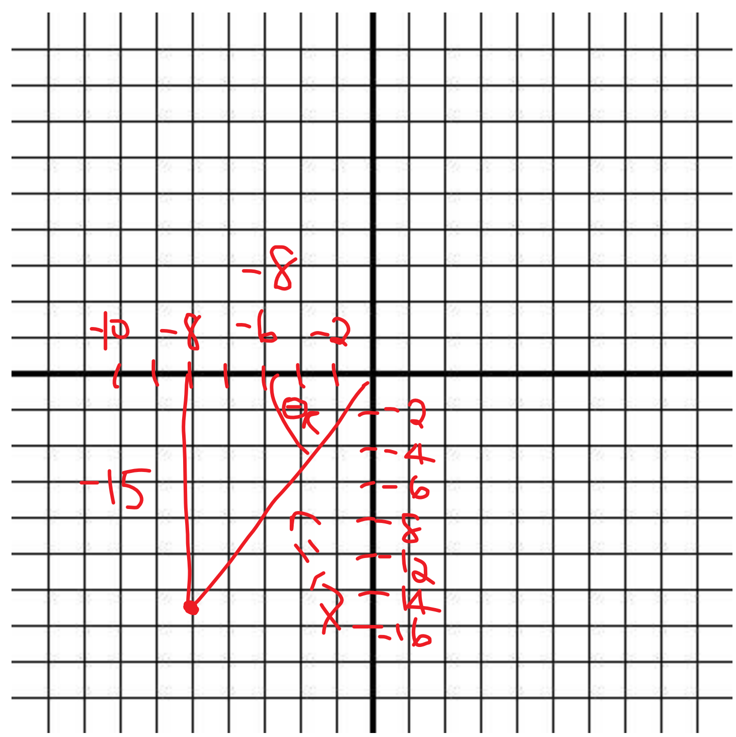
$r = \sqrt{289}$

$r = 17$

$\cos\theta = -\frac{8}{17}$

$\sin\theta = -\frac{15}{17}$

$\tan\theta = \frac{-15}{-8} = \frac{15}{8}$



Example 4:
Suppose θ is an angle in standard position with terminal arm in quadrant III and $\cos\theta = -\frac{3}{4}$. What are the exact values of sinθ and tanθ?

$\cos\theta = \frac{-3}{4}$ $\frac{a}{h}$

$x^2 + y^2 = r^2$

$(-3)^2 + y^2 = (4)^2$

$9 + y^2 = 16$

$y^2 = 16 - 9$

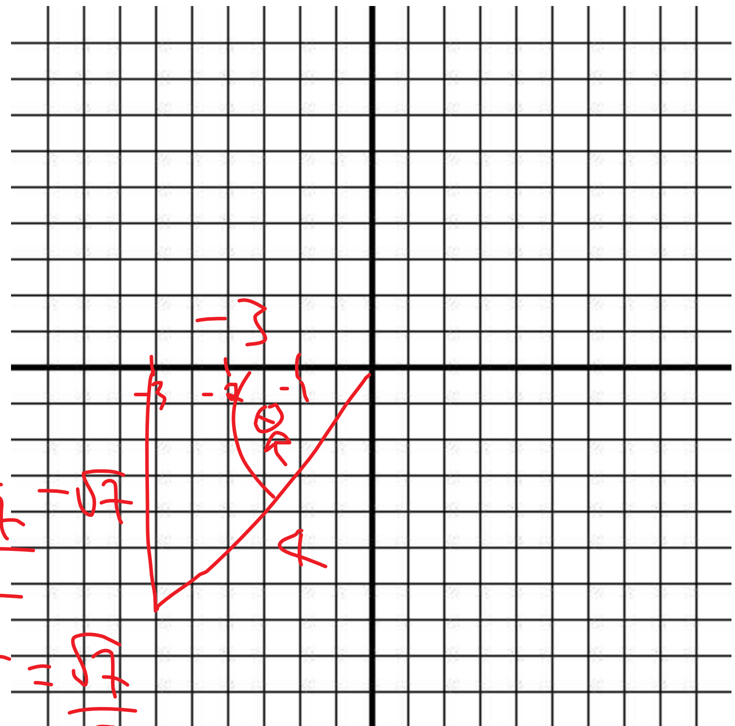
$y^2 = 7$

$\sqrt{y^2} = \sqrt{7}$

$y = \sqrt{7}$

$\sin\theta = \frac{-\sqrt{7}}{4}$

$\tan\theta = \frac{-\sqrt{7}}{-3} = \frac{\sqrt{7}}{3}$

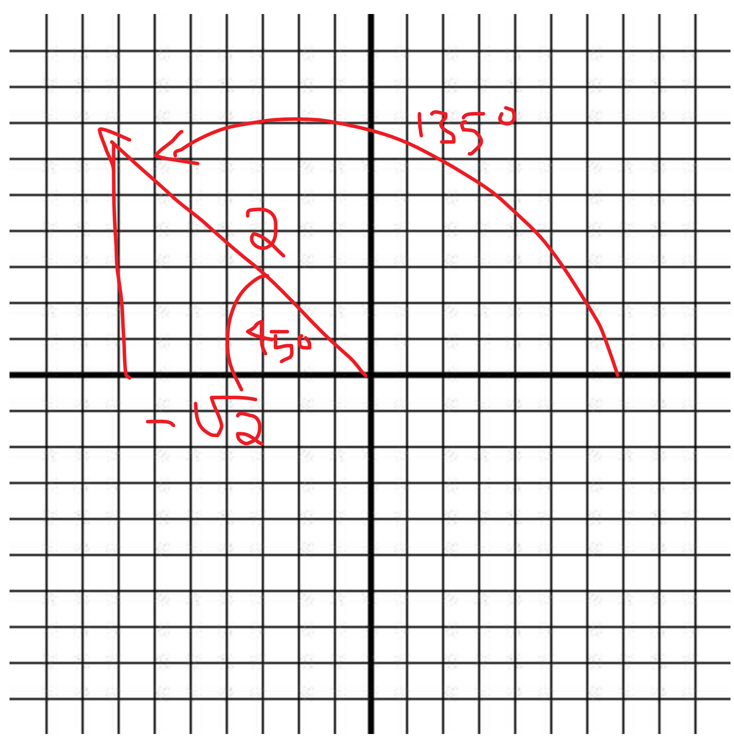


Example 5:
Determine the exact value of cos 135°.

$\theta_R = 180° - 135° = 45°$

$\cos 45° = \frac{\sqrt{2}}{2}$ $\frac{adj}{hyp}$

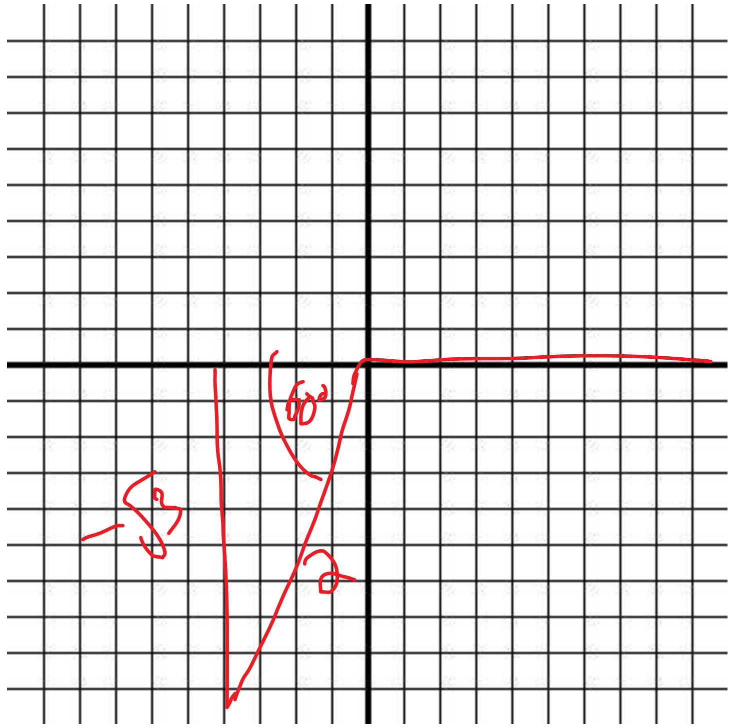
$\cos 135° = -\frac{\sqrt{2}}{2}$



Example 6:
Determine the exact value of sin 240°.

$\sin 60^\circ = \frac{\sqrt{3}}{2}$ $\frac{opp}{hyp}$

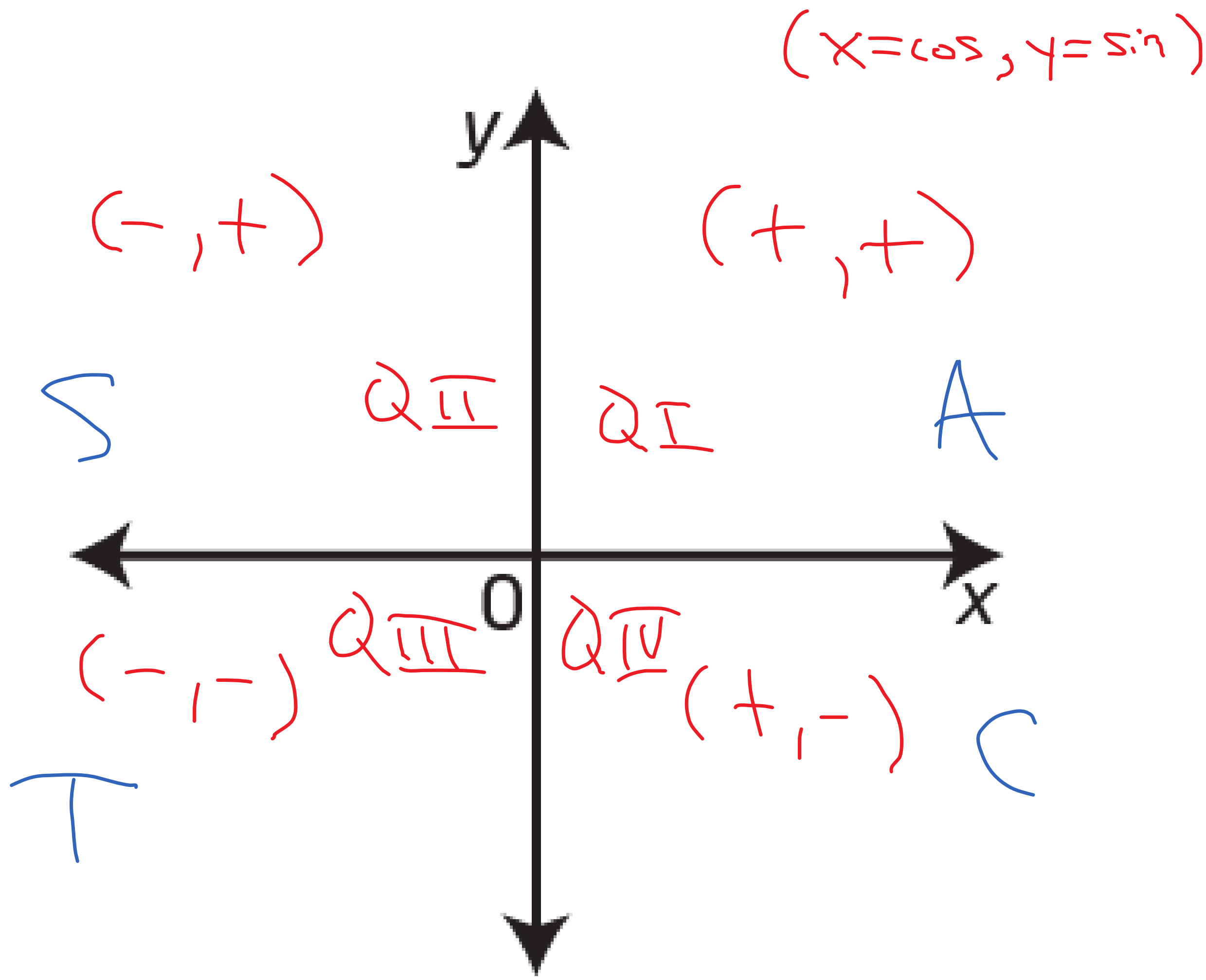
$\sin 240^\circ = -\frac{\sqrt{3}}{2}$



This process can be simplified to make it much easier to understand ~~what~~ the underlying idea. Ignoring the radius, r, notice that cosine corresponds only to values on the x-axis and sine corresponds only to values on the y-axis. So it's best to simply think:

$$x = \cos$$
$$y = \sin$$

So any quadrant where the x-axis is positive, all cosine ratios will be positive. Any quadrant where the x-axis is negative, all cosine ratios will be negative. The same holds true for the y-axis and sine.



Complete the following table. For the tanθ column, remember from Section 2.1 that:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

Quadrant	cos θ	sin θ	tan θ
I	+	+	$\frac{+}{+} = +$
II	−	+	$\frac{+}{-} = -$
III	−	−	$\frac{-}{-} = +$
IV	+	−	$\frac{-}{+} = -$

Solving for Angles Given the Sine, Cosine or Tangent Ratios

Step 1: Determine which quadrants the solution(s) will be in by looking at the sign, + or −, of the given ratio.

Step 2: Solve for the reference angle.

Step 3: Determine the measure of the related angle in standard position. The use of a diagram may be very useful here.

Example 7:
Solve for θ.

x = cos y = sin

(A) $\sin\theta = 0.5,\ 0^{o} \leq \theta \leq 360^{o}$

$\theta_R = \sin^{-1}(0.5)$

$\theta_R = 30^\circ$

QI $\theta = \theta_R \therefore \theta = 30^\circ$

QII $\theta = 180^\circ - \theta_R = 180^\circ - 30^\circ = 150^\circ$

− + ✓	+ + ✓
− −	+ −

(B) $\cos\theta = -\frac{\sqrt{3}}{2},\ 0^{o} \leq \theta \leq 360^{o}$

$\theta_R = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\theta_R = 30^\circ$

QII $\theta = 180^\circ - \theta_R = 180^\circ - 30^\circ = 150^\circ$

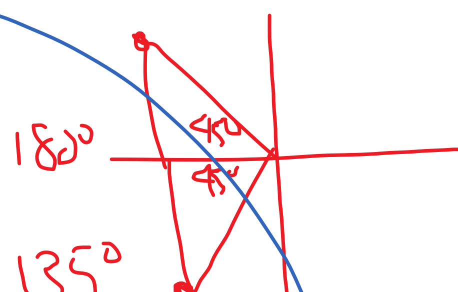
QIII $\theta = 180^\circ + \theta_R = 180^\circ + 30^\circ = 210^\circ$

− + ✓	+ +
− − ✓	+ −

(C) $\cos\theta = -\frac{1}{\sqrt{2}},\ 0^{o} \leq \theta \leq 180^{o}$

$\theta_R = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\theta_R = 45^\circ$



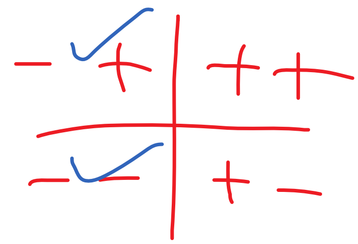
− + ✓	+ +
− − ✓	+ −

QII $= 180^\circ - 45^\circ = 135^\circ$

~~QIII $= 180^\circ + 45^\circ = 225^\circ$~~

Example 8:
Given $\cos\theta = -0.6753$, where $0^o \leq \theta \leq 360^o$, determine the measure of θ, to the nearest tenth of a degree.

$\theta_R = \cos^{-1}(0.6753)$



$\theta_R = 48^\circ$

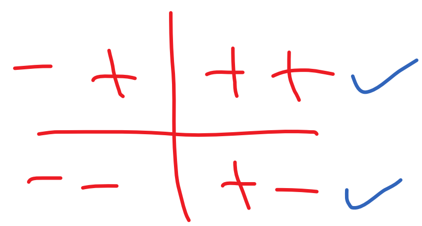
$\theta = 180^\circ - 48^\circ = 132^\circ$

$\theta = 180^\circ + 48^\circ = 228^\circ$

Example 9:
Determine the measure of θ to the nearest degree for each ratio, where $0^o \leq \theta \leq 360^o$.

(A) $\cos\theta = 0.9912$

$\theta_R = \cos^{-1}(0.9912)$



$\theta_R = 8^\circ$

QI: $\theta = \theta_R$ $\theta = 8^\circ$

QIV: $\theta = 360^\circ - 8^\circ = 352^\circ$

(B) $\sin\theta = -0.3781$

$\theta_R = \sin^{-1}(0.3781) = 22^\circ$

$\theta = 180^\circ + 22^\circ = 202^\circ$

$\theta = 360^\circ - 22^\circ = 338^\circ$

(C) $\cos\theta = -0.0746$

$\theta_R = \cos^{-1}(0.0746) = 86^\circ$

$\theta = 180^\circ - 86^\circ = 94^\circ$

$\theta = 180^\circ + 86^\circ = 266^\circ$

(D) $\sin\theta = 0.4557$

$\theta_R = \sin^{-1}(0.4557) = 27^\circ$

$\theta = \theta_R = 27^\circ$

$\theta = 180^\circ - 27^\circ = 153^\circ$

Textbook Questions: page: 96 - 99; # 1, 2, 3, 4, 5 (a), (b), 6, 7, 9, 16, 18, 22