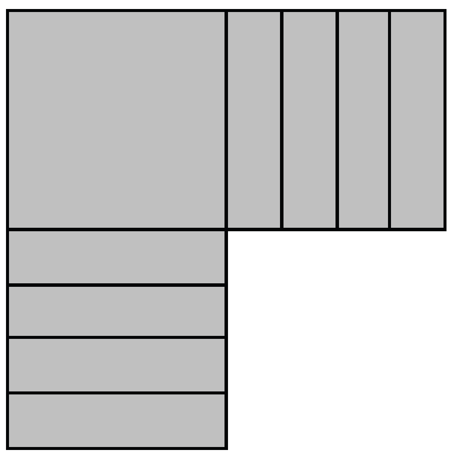
Math 2200 **Date:**____________________

3.3 Completing the Square

Perfect Square Trinomials
Recall from Mathematics 1201, a perfect square trinomial (PST) is the quadratic that results from squaring a binomial. Therefore factoring a PST involves simply taking the square root of a^2 and c^2 to determine the coefficients of both binomials that are the said factors. Fortunately when completing the square, we only deal with quadratics that have $a = 1$, or Type I Trinomials.

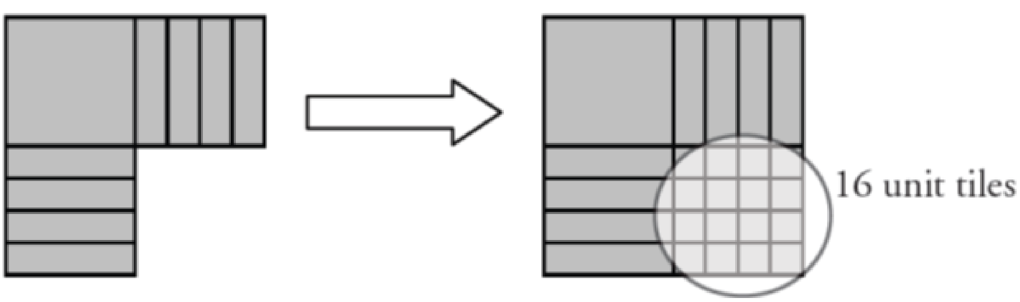
Algebra tiles can be used to visualize how a perfect square trinomial can be formed. Consider the following example. Let's model $x^2 + 8x$. The goal is to find a number, c, to create a square.



Why do the number of tiles have to be split evenly?

We need to build a square.

What tiles must be added to complete the square?



What is the expression that represents the new completed square?

$${}_{a}x^2 + \underset{b}{8}x + \underset{c}{16}$$

What is the relationship between the coefficient of the linear term and the constant term?

$16 = \left(\frac{8}{2}\right)^2$ or $c = \left(\frac{b}{2}\right)^2 \leftarrow$ Key to completing the square Not provided.

What is the trinomial written as the square of a binomial?

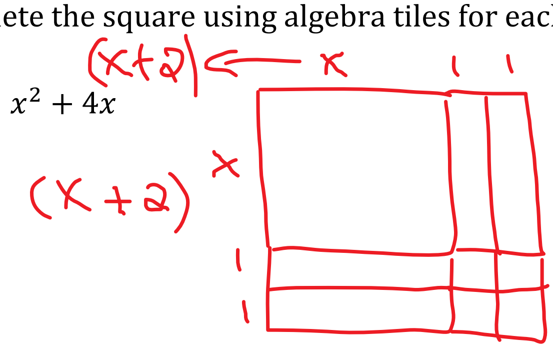
$$x^2 + 8x + 16$$
$$= (x+4)(x+4)$$
$$= (x+4)^2$$

16
1, 16
2, 8
4, 4

Example 1:

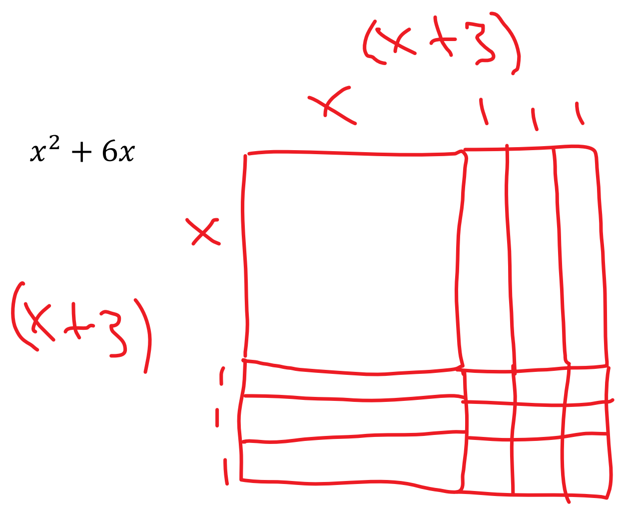
Complete the square using algebra tiles for each quadratic expression:

(A) $x^2 + 4x$



$$x^2 + 4x + 4$$
$$= (x+2)(x+2)$$
$$= (x+2)^2$$

(B) $x^2 + 6x$



$$x^2 + 6x + 9$$
$$= (x+3)(x+3)$$
$$= (x+3)^2$$

Completing the Square Algebraically

We will now move the discussion over to completing the square without using algebra tiles. Algebra tiles, while providing a great visual representation are not always practical and much more time consuming.

Example 2:

Expand each expression.

(A) $(x + 3)^2$

$= (x+3)(x+3)$

$= x(x+3) + 3(x+3)$

$= x^2 + 3x + 3x + 9$

$= x^2 + 6x + 9$

(B) $(2x - 5)^2$

$= 4x^2 - 20x + 25$

Note: What do you notice about the relationship between *a, b* and *c* and the coefficients of the binomial factors?

$2^2 = 4 \leftarrow a$

$2(2 \times 5) = 20 \leftarrow b$

$5^2 = 25 \leftarrow c$

Example 3:
Factor each perfect square trinomial.

(A) $x^2 + 10x + 25$

$\sqrt{1} = 1$ $\quad$ $\sqrt{25} = 5$

$2(1 \times 5) = 10$

PST

$(x+5)(x+5)$

(B) $4x^2 + 12x + 9$

$\sqrt{4} = 2$ $\quad$ $\sqrt{9} = 3$

$2(2 \times 3) = 12$

PST

$(2x+3)^2$

(C) $49x^2 - 28x + 4$

$\sqrt{49} = 7$ $\quad$ $\sqrt{4} = 2$

$2(7 \times 2) = 28$

PST

$(7x-2)^2$

(D) $x^2 + 16x + 49$

$\sqrt{1} = 1$ $\quad$ $\sqrt{49} = 7$

$2(1 \times 7) = 14 \neq 16$

$\therefore$ Not a PST

Note: Again, what do you notice about the relationship between *a, b* and *c* and the coefficients of the binomial factors?

With respect to (A), for all Type I Trinomials, $a=1$, the factor is $\frac{1}{2}b$.

Completing the Square
This process involves adding a value to and then subtracting the same value from a quadratic polynomial so that it contains a perfect square trinomial. This is a very useful technique in mathematics used to manipulate formulae. It is commonly called **adding zero**.

To find a value for c that makes the trinomial a perfect square, we use the formula:

$$c = \left(\frac{b}{2}\right)^2$$

You can then factor and rewrite this trinomial as the square of a binomial. In other words, rewrite it in vertex form. Note that the numeric term of the binomial factor is always $\frac{b}{2}$.

Example 4:
Complete the square by finding which value for c makes the quadratic a PST.

(A) $x^2 + 6x + c$

$c = \left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = (3)^2 = 9$

$x^2 + 6x + 9$
$= (x+3)(x+3)$
$= (x+3)^2$

(B) $x^2 + 12x + c$

$c = \left(\frac{12}{2}\right)^2 = (6)^2 = 36$

$= x^2 + 12x + 36$
$= (x+6)^2$

(C) $x^2 - 4x + c$

$c = \left(\frac{-4}{2}\right)^2 = 4$

$= x^2 - 4x + 4$
$= (x-2)^2$

(D) $x^2 + 5x + c$

$c = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$

$= x^2 + 5x + \frac{25}{4}$
$= \left(x + \frac{5}{2}\right)^2$

(E) $x^2 - 7x + c$

$= x^2 - 7x + \frac{49}{4}$

$= (x - \frac{7}{2})^2$

(F) $4x^2 + 16x$ Factor numerical GCF

$= 4(x^2 + 4x)$

$= 4(x^2 + 4x + 4)$

$= 4(x+2)^2$

Changing from Standard to Vertex Form

$$f(x) = ax^2 + bx + c \longrightarrow f(x) = a(x-p)^2 + q$$

Example 5:
Change from standard form to vertex form.

$$f(x) = x^2 - 8x + 5$$

Step 1: Isolate the first two terms (variables)

$y = (x^2 - 8x) + 5$

Step 2: Complete the square and "add 0".

$y = (x^2 - 8x + 16) + 5 - 16$

Step 3: Factor PST and simplify.

$y = (x-4)^2 - 11$

vertex: $(4, -11)$

Notice in that last example $a = 1$. When $a \neq 1$, the process is a little more complicated as you have to factor a out of $ax^2 + bx$ before you can complete the square.

Example 6:
Change from standard form to vertex form.

$$f(x) = 5x^2 + 30x + 41$$

$Y = (5x^2 + 30x) + 41$

$Y = 5(x^2 + 6x) + 41$

$Y = 5(x^2 + 6x + 9) + 41 - 45$

$Y = 5(x+3)^2 - 4$

Vertex: $(-3, -4)$

Example 7:
Your turn. Change from standard form to vertex form. Homework.

$$f(x) = 3x^2 - 12x - 9$$

$Y = 3(x^2 - 4x) - 9$

$Y = 3(x^2 - 4x + 4) - 9 - 12$

$Y = 3(x-2)^2 - 21$

Vertex: $(2, -21)$

Example 8:
Change from standard form to vertex form.

$$f(x) = -4x^2 - 16x - 9$$

$Y = (-4x^2 - 16x) - 9$

$Y = -4(x^2 + 4x) - 9$

$Y = -4(x^2 + 4x + 4) - 9 + 16$

$Y = -4(x+2)^2 + 7$

Vertex $(-2, 7)$

Example 9:
Determine the vertex of the following quadratic function: Homework

$$f(x) = 4x^2 - 28x - 23$$

$Y = 4(x^2 - 7x) - 23$

$Y = 4(x^2 - 7x + \frac{49}{4}) - 23 - 49$

$Y = 4(x - \frac{7}{2})^2 - 72$

Vertex: $(\frac{7}{2}, -72)$

Common Mistakes

Common errors occur when converting a quadratic function from standard form to vertex form. A quadratic function where $a \neq 1$, for example, sometimes causes difficulty for students. Consider the following example and discuss with students the possible errors that may occur:

$$y = -3x^2 + 18x - 23$$
$$y = -3(x^2 - 6x) - 23$$
$$y = -3(x^2 - 6x + 9) - 23 + 27$$
$$y = -3(x - 3)^2 + 4$$

- The common factor (-3) is not factored out from both the quadratic and linear terms.
- There is an incorrect sign on the linear term when a negative leading coefficient is factored out.
- The constant term inside the parentheses is doubled instead of squared.
- When a perfect square is created, the constant term inside the parentheses is not multiplied by the common factor to produce the compensated term.
- The perfect square trinomial is incorrectly factored.

Example 10:

Tom is trying to change a quadratic function from standard form to vertex form by completing the square. Identify in which step he makes an error:

(A) Step 1	$y = -2x^2 + 8x - 7$
(B) Step 2	$y = -2(x^2 - 4x) - 7$
(C) Step 3	$y = -2(x^2 - 4x + 4) - 7 - 4$ $+8$
(D) Step 4	$y = -2(x - 2)^2 - 11$

Maximum/Minimum Problems Solved By Completing the Square

While I do recommend solving these types of problems by finding the vertex using $p = -\frac{b}{2a}$, you can also use completing the square. Set up the problem and find the quadratic, then complete the square to identify the vertex.

Example 11:

On a forward dive, Greg's height above the water is given by $h(t) = -5t^2 + 10t + 3$, where t is time in seconds and h is height in feet.

(A) Find Greg's maximum height above the water.

$h = (-5t^2 + 10t) + 3$

$h = -5(t^2 - 2t) + 3$

$h = -5(t^2 - 2t + 1) + 3 + 5$

$h = -5(t-1)^2 + 8$

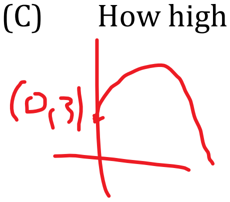
vertex: $(1, 8)$

Greg's max height is 8 ft.

(B) How long does it take him to reach that maximum height?

It takes 1s to reach the max height.

(C) How high is the diving board?

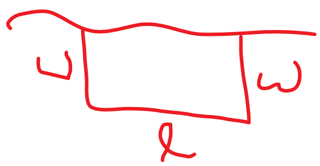


$c = 3$ Diving board is 3ft high.

(D) What is his height after 1.5 seconds?

$h(1.5) = -5(1.5)^2 + 10(1.5) + 3 = 6.75$ ft.

Example 12:
A rectangular lot is bounded on one side by a river and on the other three sides by 80 m of fencing. Find the dimensions that will enclose the maximum area.



① $2w + l = 80$

② $A = l \cdot w$

$l = -2w + 80$

$A = (-2w + 80) \cdot w$

$A = -2w^2 + 80w$

$A = -2(w^2 - 40w)$

$A = -2(w^2 - 40w + 400) + 800$

$\rightarrow A = -2(w - 20)^2 + 800$

Vertex: $(20, 800)$

$w = \underline{20m}$

Max Area $= \underline{800m^2}$

$l = -2(20) + 80$

$l = \underline{40m}$

Textbook Questions: page 192-197; # 1, 2, 3, 4, 6 b c, 8, 11, 14, 15, 16, 21, 22, 23, 27, 28