Math 2200 **Date:**______________

4.4B The Quadratic Formula

The method used for **Completing the Square** can help us to develop the **Quadratic Formula** - an extremely important formula that can help us to solve any quadratic equation.

$$(ax^2 + bx) + c = 0$$

$$a\left(x^2 + \frac{b}{a}x\right) + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \cancel{a} \cdot \frac{b^2}{4a\cancel{^2}} = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - \frac{c}{1} \cdot \frac{4a}{4a}$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - \frac{4ac}{4a}$$

$$\frac{1}{\cancel{a}} \cdot \cancel{a}\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a} \cdot \frac{1}{a}$$

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The **Quadratic Formula** is defined as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1: a b c

Find the roots of $y = 1x^2 + 10x + 21$ using the quadratic formula.

$$x = \frac{-10 \pm \sqrt{10^2 - 4(1)(21)}}{2(1)}$$

$$x = \frac{-10 \pm \sqrt{100 - 84}}{2}$$

$$x = \frac{-10 \pm \sqrt{16}}{2}$$

$$x = \frac{-10 \pm 4}{2}$$

$$x = \frac{-10 + 4}{2}, \quad x = \frac{-10 - 4}{2}$$

$$x = -\frac{6}{2}, \quad x = -\frac{14}{2}$$

$$x = -3 \qquad x = -7$$

$x^2 + 10x + 21 = 0$

$(x + 3)(x + 7) = 0$

$x = -3, x = -7$

Example 2:
Solve using the quadratic formula, express your answer as an exact value.

$$-7x - 1 = -5x^2$$

$$5x^2 - 7x - 1 = 0$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(5)(-1)}}{2(5)}$$

$$x = \frac{7 \pm \sqrt{49 + 20}}{10}$$

$$x = \frac{7 \pm \sqrt{69}}{10}$$

$$x = \frac{7 + \sqrt{69}}{10}, \quad x = \frac{7 - \sqrt{69}}{10}$$

Example 3:
Solve using the quadratic formula, express your answer to the nearest hundredth.

$$x(3x+5)=2$$

$$3x^2+5x-2=0$$

$$x=\frac{-5\pm\sqrt{5^2-4(3)(-2)}}{2(3)}$$

$$x=\frac{-5\pm\sqrt{25+24}}{6}$$

$$x=\frac{-5\pm\sqrt{49}}{6}$$

$$x=\frac{-5\pm 7}{6}$$

$$x=\frac{-5+7}{6},\quad x=\frac{-5-7}{6}$$

$$x=\frac{2}{6},\quad x=-\frac{12}{6}$$

$$x=\frac{1}{3}\qquad x=-2$$

$$x=0.33\qquad x=-2.00$$

Example 4:
Solve using the quadratic formula, express your answer to the nearest hundredth.

$$-4 + 3x^2 = x$$

$3x^2 - x - 4 = 0$

$x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-4)}}{2(3)}$

$x = \dfrac{1 \pm \sqrt{1 + 48}}{6}$

$x = \dfrac{1 \pm \sqrt{49}}{6} \longrightarrow x = \dfrac{1 \pm 7}{6}$

$x = \dfrac{1+7}{6}, \quad x = \dfrac{1-7}{6}$

$x = \dfrac{8}{6}, \quad x = \dfrac{-6}{6}$

$x = 1.33, \quad x = -1.00$

Example 5

Mary, David, and Ron are students in a group. They are given the equation $A = x^2 + 3x - 110$ where A represents the area of a field and x represents the width in metres. The students were asked to find the width if the area was 100 m^2. Each student decided to solve the equation using their own preferred method. Here are their solutions:

Mary

$$x^2 + 3x - 110 = 100$$

$$x^2 + 3x - 210 = 0$$

$$x = \frac{-3 \pm \sqrt{9 - (4)(1)(-210)}}{2}$$

$$x = \frac{3 \pm \sqrt{-831}}{2}$$

849

$$x = \frac{3 \pm 28.827}{2}$$

$$x = 15.9 \text{ or } x = -12.9$$

width is 15.9 m

David

$$x^2 + 3x - 110 = 100$$

$$x^2 + 3x - 10 = 0$$

-210

$$(x + 5)(x - 3) = 0$$

$$x = -5 \text{ or } x = 3$$

width is 3 m

Ron

100

$$x^2 + 3x - 110 = 0$$

$$x = \frac{-3 \pm \sqrt{9 - (4)(1)(-110)}}{2}$$

$$x = \frac{-3 \pm \sqrt{9 + 440}}{2}$$

$$x = \frac{-3 \pm \sqrt{449}}{2}$$

$$x = \frac{-3 \pm 21.2}{2}$$

$$x = 9.1 \text{ or } x = -12.1$$

width is 9.1 m

Identify and explain the errors in the students' work, then write the correct solution.

$$x = \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(-210)}}{2(1)}$$

$$x = \frac{-3 \pm \sqrt{849}}{2}$$

$$x = \frac{-3 \pm 29.1}{2} \rightarrow x = \frac{-3 + 29.1}{2}, \quad x = \frac{-3 - 29.1}{2}$$

$$x = 13.05, \quad x = -\cancel{16}.05$$

Example 6

Jonas wants to frame an oil original painted on canvas measuring 50 cm by 60 cm. Before framing, he places the painting on a rectangular mat so that a uniform strip of the mat shows on all sides of the painting. The area of the mat is twice the area of the painting. How wide is the strip of exposed mat showing on all sides of the painting, to the nearest tenth of a centimetre.

$l \cdot w = A$

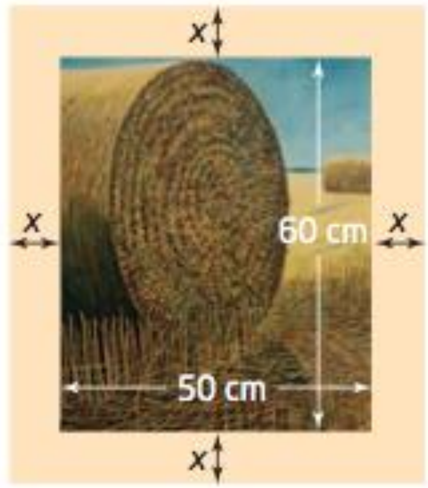
$A_p = 50 \times 60$

$= 3000$

$l \cdot w = 6000$

$A_T = 2(3000)$

$= 6000$



$2x+60$

$2x+50$

$(2x+50)(2x+60) = 6000$

$4x^2 + 120x + 100x + 3000 - 6000 = 0$

$$\frac{4x^2 + 220x - 3000}{4} = \frac{0}{4}$$

$x^2 + 55x - 750 = 0$

$$x = \frac{-55 \pm \sqrt{55^2 - 4(1)(-750)}}{2(1)}$$

$$x = \frac{-55 \pm \sqrt{6025}}{2}$$

$$x = \frac{-55 \pm 77.6}{2}$$

$$x = \frac{-55 + 77.6}{2}$$

$x = 11.3$

or

$$x = \frac{-55 - 77.6}{2}$$

$x = \cancel{-66.3}$

x is 11.3 cm

Example 7
Find two consecutive, positive integers such that the sum of their squares is 265.

$x, x+1$

$x^2+(x+1)^2=265$

$x^2+x^2+2x+1-265=0$

$\frac{2x^2+2x-264}{2}=\frac{0}{2}$

$x^2+x-132=0$

$x=\frac{-1\pm\sqrt{1^2-4(1)(-132)}}{2(1)}$

$x=\frac{-1\pm\sqrt{529}}{2}$

$x=\frac{-1\pm 23}{2}$

$x=\frac{-1+23}{2}, x=\frac{-1-23}{2}$

$x=\frac{22}{2}, x=\frac{-24}{2}$

$x=11, \cancel{x=-12}$

$x+1=11+1=12$

Imaginary Roots

Originally coined in the 17th century as a derogatory term and regarded as fictitious or useless, the concept gained wide acceptance following the work of Leonhard Euler and Carl Friedrich Gauss.

For Math 2200, we are only concerned with imaginary numbers as they are used to represent the roots of quadratics that do not cross the x-intercept. When using the quadratic formula to solve quadratic equations, we simply incorporate the fact that $i = \sqrt{-1}$ as we did in section 4.4A.

Example 8:

Solve using the quadratic formula:

(A) $\overset{a}{2}x^2 + \overset{b}{3}x + \overset{c}{5} = 0$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(5)}}{2(1)}$$

$$x = \frac{-3 \pm \sqrt{9-40}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{-31}}{4}$$

$$x = \frac{-3 \pm \sqrt{31}\sqrt{-1}}{4}$$

$$x = \frac{-3 \pm \sqrt{31}\,i}{4}$$

$$x = \frac{-3 + \sqrt{31}\,i}{4}, \quad x = \frac{-3 - \sqrt{31}\,i}{4}$$

(B) $x^2 + 2x + 5 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$x = \frac{-2 \pm \sqrt{-16}}{2}$$

$$x = \frac{-2 \pm \sqrt{16}\sqrt{-1}}{2}$$

$$x = \frac{-\cancel{2} \pm \cancel{4}i}{\cancel{2}}$$

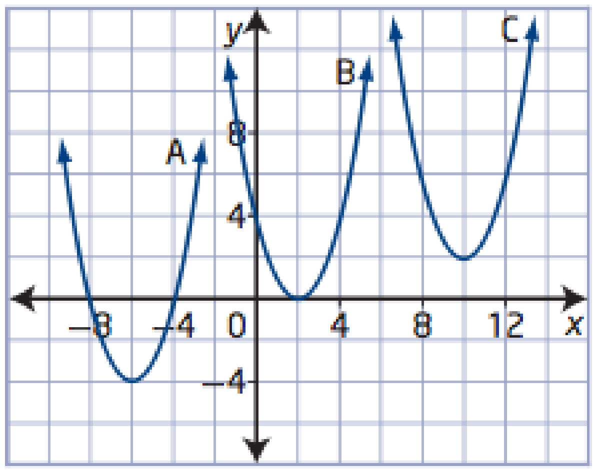
$$x = -1 \pm 2i$$

$$x = -1 + 2i, \quad x = -1 - 2i$$

Discriminant
The **discriminant** is the radicand part of the quadratic equation:

$$D = b^2 - 4ac.$$

It is a useful tool if only the nature of the roots are required as opposed to the actual root.



- If $b^2 - 4ac > 0$ then there are **two** real roots, as shown by graph A.
- If $b^2 - 4ac = 0$ then there is **one** real root, as shown by graph B.
- If $b^2 - 4ac < 0$ then there are **no** real roots, as shown by graph C.

Example 8
Use the discriminant to determine the nature of the roots for $3x^2 - 5x = -9$.

$3x^2 - 5x + 9 = 0$
$a \quad b \quad c$

$D = b^2 - 4ac$

$D = (-5)^2 - 4(3)(9)$

$= 25 - 108$

$= -83$

$-83 < 0$

$\therefore$ No real roots

2 Imaginary roots

Example 9

How many roots does the following equation have?

$$0.25x^2 - 3x + 9 = 0$$

$D = b^2 - 4ac$

$= (-3)^2 - 4(0.25)(9)$

$= 9 - 9$

$= 0$ $\therefore$ 1 real root

Example 10

When Chantal was asked to describe the roots of the equation $14x^2 - 5x = -5$, she rearranged the equation so that it would equal zero, then used the quadratic formula to find the roots. Her workings are shown below. Edward said that she didn't have to do all that and he then showed the class his work. Are they both correct. Identify your preferred.

Chantal	Edward
$x = \frac{5 \pm \sqrt{25-280}}{28}$	$b^2 - 4ac$
$x = \frac{5 \pm \sqrt{-255}}{28}$	$= 25 - 280$
$x = \frac{5 \pm \sqrt{-1}\sqrt{255}}{28}$	$= -255$
$x = \frac{5 \pm i\sqrt{255}}{28}$	no real roots
imaginary roots	

Both are correct. Edward's solution is more efficient.

Textbook Questions: page 254 – 257 #1, 2, 3, 4, 7, 8, 9, 10, 11, 13, 14, 15, 21