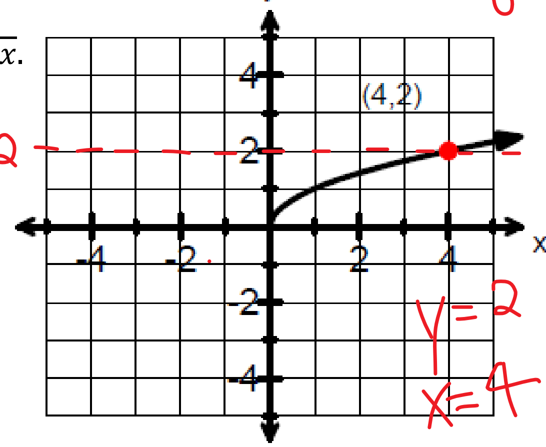
Math 2200

5.3 Radical Equations

Restrictions

If a radical represents a real number and has an even index, the radicand must be non-negative. Given $\sqrt[n]{x}$, if n is even, the radicand is non-negative.

Lets take a look at the graph of the radical function $y = \sqrt{x}$.



$y=2$

$y=2$
$x=4$

Focusing on the point $(4, 2)$, how would you algebraically solve the equation given only the y-coordinate 2?

$y=\sqrt{x}$, $y=2$

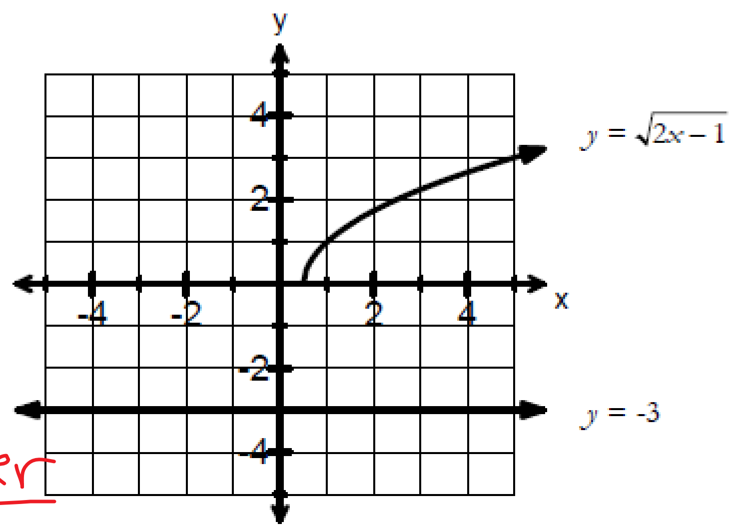
$y=y$ $\rightarrow x=4$

$\sqrt{x}=2$

$(\sqrt{x})^2=(2)^2$

Now consider the radical equation $\sqrt{2x-1} \neq -3$:

$y=\sqrt{2x-1}$, $y=-3$



No solution.

The square root of an expression can <u>never</u> be negative.

Example 1:
Using Desmos Graphing, and then algebraically, explain why the domain of $\sqrt{2x-5}$ is $x \geq \frac{5}{2}$, while the domain of $\frac{1}{\sqrt{2x-5}}$ is $x > \frac{5}{2}$.

$y = \sqrt{2x-5}$

$2x-5 \geq 0$

$2x \geq 5$

$\frac{2x}{2} \geq \frac{5}{2}$

$x \geq 5/2$

$y = \frac{1}{\sqrt{2x-5}}$

$2x-5 > 0$

$2x > 5$

$\frac{2x}{2} > \frac{5}{2}$

$x > 5/2$

Example 2:
What are the restrictions for each radical?

(A) $\sqrt{x+6}$

$x+6 \geq 0$

$x \geq -6$

(B) $\sqrt{x-11}$

$x \geq 11$

(C) $\sqrt{2x+8}$

$2x+8 \geq 0$

$2x \geq -8$

$\frac{2x}{2} \geq \frac{-8}{2}$

$x \geq -4$

A **radical** equation is an equation with radicals that have variables in the radicands. When solving a radical equation, remember to:

- identify any restrictions on the variable
- identify whether any roots are extraneous by determining whether the values satisfy the original equation.

Example 3:
State the restrictions and Solve.

(A) $\sqrt{4x} = 8$

$4x \geq 0$

$\frac{4x}{4} \geq \frac{0}{4}$

$x \geq 0$

$(\sqrt{4x})^2 = 8^2$

$4x = 64$

$\frac{4x}{4} = \frac{64}{4}$

$x = 16$

Check: $\sqrt{4(16)} = 8$

$\sqrt{64} = 8$

$8 = 8$ ✓

(B) $\sqrt{x+4} = 5$

$x + 4 \geq 0$

$x \geq -4$

$(\sqrt{x+4})^2 = (5)^2$

$x + 4 = 25$

$x = 25 - 4$

$x = 21$

Check:

$\sqrt{21+4} = 5$

$\sqrt{25} = 5$

$5 = 5$ ✓

(C) $\sqrt{2x-3} \neq -2$

Working With Inequalities

- The same number can be added to both sides of an inequality.
- The same number can be subtracted from both sides of an inequality.
- Both sides of an inequality can be multiplied or divided by the same positive number.
- If an inequality is multiplied or divided by a negative number, then: $>$ becomes $<$

$\geq$ becomes $\leq$

$<$ becomes $>$

$\leq$ becomes $\geq$

Example 4:
Solve each inequality for x:

(A) $-7x \geq 98$

$$\frac{-7x}{-7} \geq \frac{98}{-7}$$

$$x \leq -14$$

why?

$$-7x \geq 98$$

$$\frac{-98}{7} \geq \frac{7x}{7}$$

$$-14 \geq x$$

or

$$x \leq -14$$

(B) $-12x + 4 \leq 0$

$$\frac{-12x}{-12} \leq \frac{-4}{-12}$$

$$x \geq \frac{1}{3}$$

Example 5:

What is the value of x in the equation $\sqrt{-3x+6} = 6$?

(A) -10
(B) -2
(C) 0
(D) 10

~~Explain your reasoning:~~

$(\sqrt{-3x+6})^2 = 6^2$

$-3x+6=36$

$-3x=36-6$

$\frac{-3x}{-3}=\frac{30}{-3}$

$x=-10$

$-3x+6\geq 0$

$\frac{-3x}{-3}\geq\frac{-6}{-3}$

$x\leq 2$

Rather than squaring both sides of a radical equation, students sometimes mistakenly square the individual terms. When solving $3+\sqrt{2x+1}=7$, they may not isolate the radical. Squaring each term results in the incorrect equation, for example:

$$3+\sqrt{2x+1}=7$$

$$\cancel{(3)^2+\left(\sqrt{2x+1}\right)^2=(7)^2}$$

$(3+\sqrt{2x+1})^2=7^2$

$(3+\sqrt{2x+1})(3+\sqrt{2x+1})=49 \leftarrow$ Crazy!

Example 6:
State any restrictions and solve for x.

* Isolate the radical!

(A) $3 + \sqrt{2x+1} = 7$

$\sqrt{2x+1} = 7 - 3$

$\sqrt{2x+1} = 4$

$\left(\sqrt{2x+1}\right)^2 = 4^2$

$2x + 1 = 16$

$2x = 16 - 1$

$\frac{2x}{2} = \frac{15}{2}$

$x = 15/2$

$2x + 1 \geq 0$

$\frac{2x}{2} \geq \frac{-1}{2}$

$x \geq -1/2$

Check:

$3 + \sqrt{\cancel{2}\left(\frac{15}{\cancel{2}}\right) + 1} = 7$

$3 + \sqrt{15 + 1} = 7$

$3 + \sqrt{16} = 7$

$3 + 4 = 7$

$7 = 7$ ✓

(B) $\sqrt{3x-5} - 2 = 3$

$\sqrt{3x-5} = 3 + 2$

$\left(\sqrt{3x-5}\right)^2 = (5)^2$

$3x - 5 = 25$

$3x = 25 + 5$

$3x = 30$

$\frac{3x}{3} = \frac{30}{3}$

$x = 10$

$3x - 5 \geq 0$

$\frac{3x}{3} \geq \frac{5}{3}$

$x \geq 5/3$

Check:

$\sqrt{3(10) - 5} - 2 = 3$

$\sqrt{30 - 5} - 2 = 3$

$\sqrt{25} - 2 = 3$

$5 - 2 = 3$

$3 = 3$ ✓

(C) $5 + \sqrt{2x-1} = 12$

$x \geq \frac{1}{2}$

$\sqrt{2x-1} = 12 - 5$

$\left(\sqrt{2x-1}\right)^2 = (7)^2$

$2x - 1 = 49$

$2x = 50$

$\frac{2x}{2} = \frac{50}{2}$

$x = 25$

Check:

$5 + \sqrt{2(25) - 1} = 12$

$5 + \sqrt{50 - 1} = 12$

$5 + \sqrt{49} = 12$

$5 + 7 = 12$

$12 = 12$ ✓

Example 7:
Is the solution to the following radical equation correct? Justify your answer.

$$3+2\sqrt{n+4}=5$$

X $$5\sqrt{n+4}=5$$

$$\sqrt{n+4}=1$$

$$\left(\sqrt{n+4}\right)^2=1^2$$

$$n+4=1$$

$$n=-3$$

$$2\sqrt{n+4}=5-3$$

$$\frac{\cancel{2}\sqrt{n+4}}{\cancel{2}}=\frac{2}{2}$$

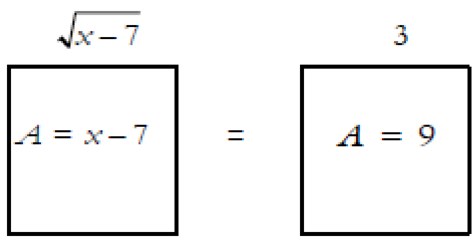
$$\sqrt{n+4}=1$$

The Area Model
The area model can also be used to solve radical equations. Recall that in Grade 8, you viewed the area of the square as the perfect square number, and the side length of the square as the square root. Recall that if a square has an area of 4, then its side has a length of 2. Similarly, if a square has an area of 3, then its side has a length of $\sqrt{3}$.

Consider the following example: Solve $\sqrt{x-7}=3$.

$x \geq 7$

$\sqrt{x-7}$



3

$$x-7=9$$

$$x=9+7$$

$$x=16$$

Check:

$$\sqrt{16-7}=3$$

$$\sqrt{9}=3$$

$$3=3 \checkmark$$

Extraneous Roots

Extraneous roots occur because squaring both sides and solving the quadratic equation may result in roots that do not satisfy the original equation.

Example 8:

Solve and check your solution for extraneous roots:

$$n - \sqrt{5-n} = -7$$

* Isolate the radical!

$5 - n \geq 0$

$5 \geq n$

$n \leq 5$

$n + 7 = \sqrt{5-n}$

$(n+7)^2 = (\sqrt{5-n})^2$

$n^2 + 14n + 49 = 5 - n$

$n^2 + 14n + n + 49 - 5 = 0$

$n^2 + 15n + 44 = 0$

$(n+4)(n+11) = 0$

$n + 4 = 0, \quad n + 11 = 0$

$\boxed{n = -4} \quad \cancel{n = -11}$

Check: $n = -4$

$-4 - \sqrt{5-(-4)} = -7$

$-4 - \sqrt{9} = -7$

$-4 - 3 = -7$

$-7 = -7$ ✓

$n = -11$

$-11 - \sqrt{5-(-11)} = -7$

$-11 - \sqrt{16} = -7$

$-11 - 4 = -7$

$-15 \neq -7$ ✗

Example 9:

Solve and check your solution for extraneous roots:

$$x - \sqrt{2x+3} = 6$$

$2x + 3 \geq 0$

$x \geq -\frac{3}{2}$

$x - 6 = \sqrt{2x+3}$

$(x-6)^2 = (\sqrt{2x+3})^2$

$x^2 - 12x + 36 = 2x + 3$

$x^2 - 14x + 33 = 0$

$(x-3)(x-11) = 0$

$x - 3 = 0$, $x - 11 = 0$

$\cancel{x = 3}$ $\boxed{x = 11}$

Check:

$x = 3$

$3 - \sqrt{2(3)+3} = 6$

$3 - \sqrt{9} = 6$

$3 - 3 = 6$

$0 \neq 6$ ✗

$x = 11$

$11 - \sqrt{2(11)+3} = 6$

$11 - \sqrt{25} = 6$

$11 - 5 = 6$

$6 = 6$ ✓

Radical Equations With Two Radicals

When we solve equations that involve two radical expressions, we have to square both sides in an equation as with single radicals. However, the resulting equation will still contain a radical. Therefore, we need to repeat the process of isolating the radical term and squaring both sides of the equation again.

Example 10:

Solve for x. State any restrictions and extraneous roots.

$x \geq -7$, $\underline{x \geq 0}$

$$\sqrt{x+7} = \sqrt{x} + 1$$

$$(\sqrt{x+7})^2 = (\sqrt{x}+1)^2$$

$$x+7 = x + 2\sqrt{x} + 1$$

$$x - x + 7 - 1 = 2\sqrt{x}$$

$$\frac{6}{2} = \frac{2\sqrt{x}}{2}$$

$$3 = \sqrt{x}$$

$$(3)^2 = (\sqrt{x})^2$$

$$9 = x$$

or

$$x = 9$$

Check:

$$\sqrt{9+7} = \sqrt{9} + 1$$

$$\sqrt{16} = 3+1$$

$$4 = 3+1$$

$$4 = 4 \checkmark$$

Example 11:

Solve and check your solution:

$$7+\sqrt{3x}=\sqrt{5x+4}+5, x\geq 0$$

$7-5+\sqrt{3x}=\sqrt{5x+4}$

$2+\sqrt{3x}=\sqrt{5x+4}$

$(2+\sqrt{3x})^2=(\sqrt{5x+4})^2$

$4+4\sqrt{3x}+3x=5x+4$

$4\sqrt{3x}=5x-3x+4-4$

$\frac{4\sqrt{3x}}{2}=\frac{2x}{2}$

$(2\sqrt{3x})^2=(x)^2$

$4(3x)=x^2$

$12x=x^2$

$x^2-12x=0$

$x(x-12)=0$

$x=0, \quad x-12=0$

$x=12$

Check:

$x=0$

$7+\sqrt{3(0)}=\sqrt{5(0)+4}+5$

$7=\sqrt{4}+5$

$7=2+5$

$7=7$ ✓

$7+\sqrt{3(12)}=\sqrt{5(12)+4}+5$

$7+\sqrt{36}=\sqrt{64}+5$

$7+6=8+5$

$13=13$ ✓

Example 12:
Solve and check your solution:

$$\sqrt{3+j}+\sqrt{2j-1}=5, j\geq\frac{1}{2}$$

$$\sqrt{3+j}=-\sqrt{2j-1}+5$$

$$(\sqrt{3+j})^2=(-\sqrt{2j-1}+5)^2$$

$$3+j=2j-1-10\sqrt{2j-1}+25$$

$$10\sqrt{2j-1}=2j-j+24-3$$

$$(10\sqrt{2j-1})^2=(j+21)^2$$

$$100(2j-1)=j^2+42j+441$$

$$200j-100=j^2+42j+441$$

$$0=j^2-158j+541$$

$$j=\frac{-(-158)\pm\sqrt{(-158)^2-4(1)(541)}}{2(1)}$$

$$j=\frac{158\pm\sqrt{24964-2164}}{2}$$

Check: . . .

$$j=\frac{158\pm\sqrt{22800}}{2}$$

$$j=\frac{158\pm\sqrt{400}\sqrt{57}}{2}$$

$$j=\frac{158\pm20\sqrt{57}}{2}$$

$$j=79\pm10\sqrt{57}$$

Example 13:

What is the speed, in metres per second, of a 0.4 kg football that has 28.8 J of kinetic energy? Use the formula $E_k = \frac{1}{2}mv^2$, where E_k represents the kinetic energy in joules; m represents the mass in kilograms; and v represents the speed, in metres per second.

$$E_k = \frac{1}{2}mv^2$$

$$28.8 = \frac{1}{2}(0.4)v^2$$

$$\frac{28.8}{0.2} = \frac{0.2v^2}{0.2}$$

$$\sqrt{144} = \sqrt{v^2}$$

$$v = 12 m/s$$

or

$$E_k = \frac{1}{2}mv^2$$

$$2E_k = \frac{1}{\cancel{2}}mv^2 \cdot \cancel{2}$$

$$\frac{2E_k}{m} = \frac{mv^2}{m}$$

$$\sqrt{\frac{2E_k}{m}} = \sqrt{v^2}$$

$$v = \sqrt{\frac{2E_k}{m}} \rightarrow v = \sqrt{\frac{2(28.8)}{0.4}}$$

$$v = 12 m/s$$

Example 14:

Collision investigators can approximate the initial velocity, v, in kilometres per hour, of a car based on the length, l, in metres, of the skid mark. The formula $V = 12.6\sqrt{l} + 8$, where $l \geq 0$ models the relationship. What length of skid is expected if a car is travelling 50 km/hr when the brakes are applied?

$$50 = 12.6\sqrt{l} + 8$$

$$50 - 8 = 12.6\sqrt{l}$$

$$\frac{42}{12.6} = \frac{12.6\sqrt{l}}{12.6}$$

$$(3.33)^2 = (\sqrt{l})^2$$

$$l = 11.1 m$$

or

$$V = 12.6\sqrt{l} + 8$$

$$\frac{V-8}{12.6} = \frac{12.6\sqrt{l}}{12.6}$$

$$\left(\frac{V-8}{12.6}\right)^2 = (\sqrt{l})^2$$

$$l = \left(\frac{V-8}{12.6}\right)^2$$

$$l = \left(\frac{50-8}{12.6}\right)^2 = 11.1 m$$

Example 15:

The surface area, S, of a sphere with radius r can be found using the equation $S = 4\pi r^2$.

(A) Using the given equation, how could you find the radius of a sphere given its surface area? Write the equation.

$$\frac{S}{4\pi} = \frac{\cancel{4\pi} r^2}{\cancel{4\pi}}$$

$$r^2 = \frac{S}{4\pi}$$

$$\sqrt{r^2} = \sqrt{\frac{S}{4\pi}}$$

$$r = \sqrt{\frac{S}{4\pi}}$$

(B) The surface area of a ball is 426.2 cm^2. What is its radius?

$$r = \sqrt{\frac{426.2}{4\pi}} = 5.8 \text{ cm}$$

Textbook Questions: page 300-303, #1 (a), (c), 3 (c), 4, 6 (a), (c), 7, 8, 10, 12, 14, 15, 16