Math 2200

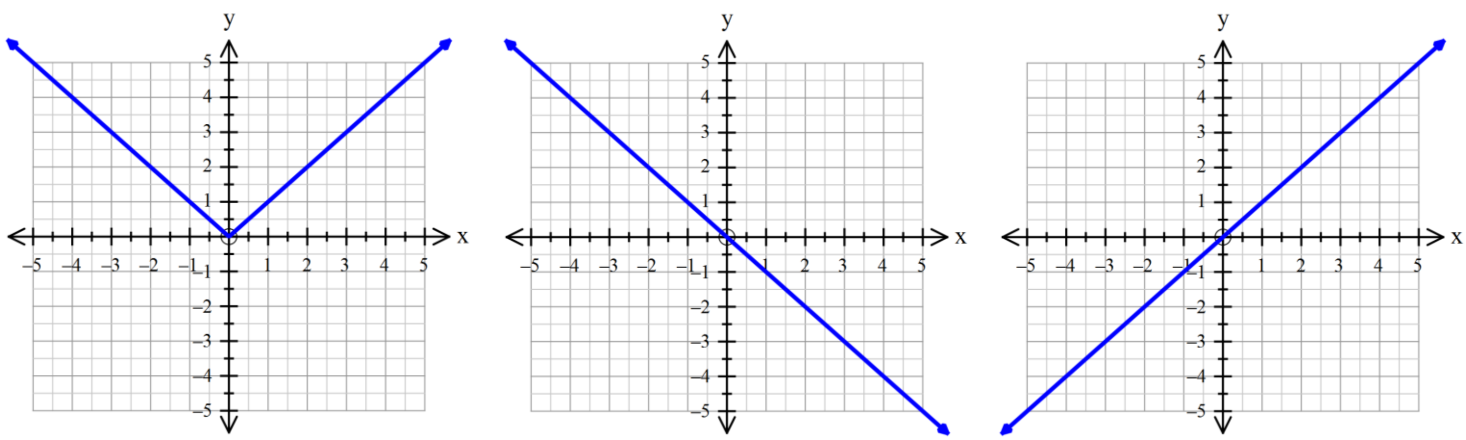
7.2 Absolute Value Functions

An **absolute value function** is a function that involves the absolute value of a variable. Basically applying the absolute value to any function makes all range values become positive.

Graphically, anything below the x-axis gets "flipped" above the x-axis.

Let's take a look at the function the graphs of $y = |x|$ and compare it with two other graphs, $y = x$ and $y = -x$

$y = |x|$ $\qquad$ $y = -x$ $\qquad$ $y = x$



Notice that the vertex, (0, 0), divides the graph of $y = |x|$ into two distinct pieces. For all values of x less than zero, the y-value is $-x$. For all values of x greater than zero, the y-value is x. So we can look at the graph of the absolute value of x as being made up of pieces of two separate graphs. We only want the parts where the range values are above zero. Therefor we can define the graph of $y = |x|$ as a piecewise function.

Piecewise Function: a function composed of two or more separate functions or pieces, each with its specific domain, that combine to define the overall function.

Using the definition of absolute value, we can describe the graph of $y = |x|$ as:

$$y = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

Why do you think it's: $-x$, if $x < 0$ and not $x \leq 0$?

If $x \leq 0$, and $x \geq 0$, we have two y-values for one x-value. By def$^{\underline{n}}$, not a function.

In general, all absolute value functions can be represented as piecewise functions:

$$y = \begin{cases} f(x), \text{ where } f(x) \geq 0 \\ -f(x), \text{ where } f(x) < 0 \end{cases}$$

Example 1:
Consider the absolute value function: $y = |2x - 4|$.

(A) Determine the y-intercept and x-intercept.

y-intercept: $x=0$

- Apply absolute value to all y-values.

$y=|2(0)-4|$
$y=|-4|$
$y=4$ $(0,4)$

x-intercepts: $y=0$

$0=2x-4$
$\frac{4}{2}=\frac{2x}{2}$
$x=2$
$(2,0)$

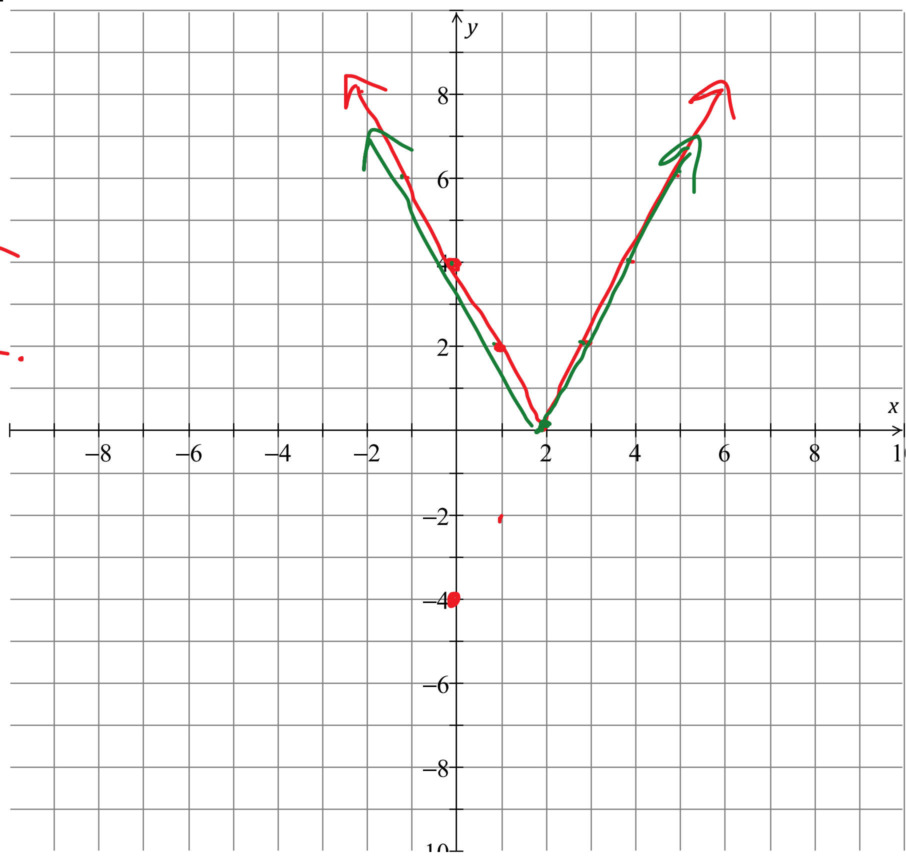
Invariant Point: a point that remains unchanged when a transformation is applied. In the case of absolute value, the x-intercept will always be an invariant point because the absolute value of 0 is always 0.

(B) Sketch the graph.

Method 1: Plot the function without absolute value, then change all negative y-values to positive.

Method 2: Apply def^n of absolute value to the function

$y=|2x-4|$
$$y=\begin{cases} 2x-4, & f(x)\geq 0 \\ -(2x-4), & f(x)<0 \end{cases}$$



(C) State the domain and range:

Domain: $\{x \mid x \in R\}$

Range: $\{y \mid y \geq 0, y \in R\}$

(D) Express as a piecewise function:

Invariant point

$$y = \begin{cases} 2x-4, & x \geq 2 \\ -(2x-4), & x < 2 \end{cases}$$

Example 2:
Graph the absolute value function: $y = |-x^2 + 2x + 8|$.

(A) Determine the y-intercept and x-intercepts (invariant points).

y-intercept: $x=0$

$y = |-(0)^2 + 2(0) + 8|$

$y = |8| = 8$

$(0, 8)$

x-intercept: $y=0$

$-x^2 + 2x + 8 = 0$

$x^2 - 2x - 8 = 0$

$(x+2)(x-4) = 0$

$x = -2, x = 4$

$\rightarrow (-2, 0), (4, 0)$

(B) Sketch the graph.

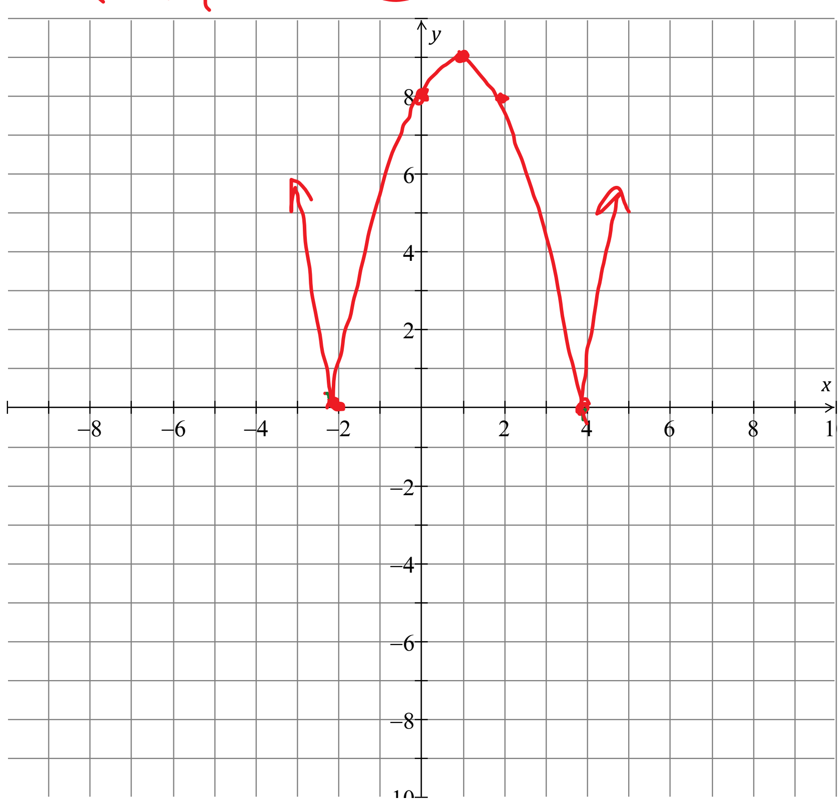
Vertex:

$p = \frac{-b}{2a} = \frac{-2}{2(-1)} = 1$

$q = |-(1)^2 + 2(1) + 8|$

$q = |9| = 9$

$(1, 9)$



(C) State the domain and range:

Domain: $\{x \mid x \in R\}$

Range: $\{y \mid y \geq 0, y \in R\}$

(D) Express as a piecewise function:

$$y = \begin{cases} -x^2+2x+8, & -2 \leq x \leq 4 \\ -(-x^2+2x+8), & x < -2, x > 4 \end{cases}$$

Example 3:
Consider the absolute value function: $y = |x^2 - x - 2|$.

(A) Determine the y-intercept and x-intercepts.

y-intercept: $x=0$

$y = |-(0)^2 - 0 - 2|$

$y = |-2|$

$y = 2$

$(0,2)$

x-intercept: $y=0$

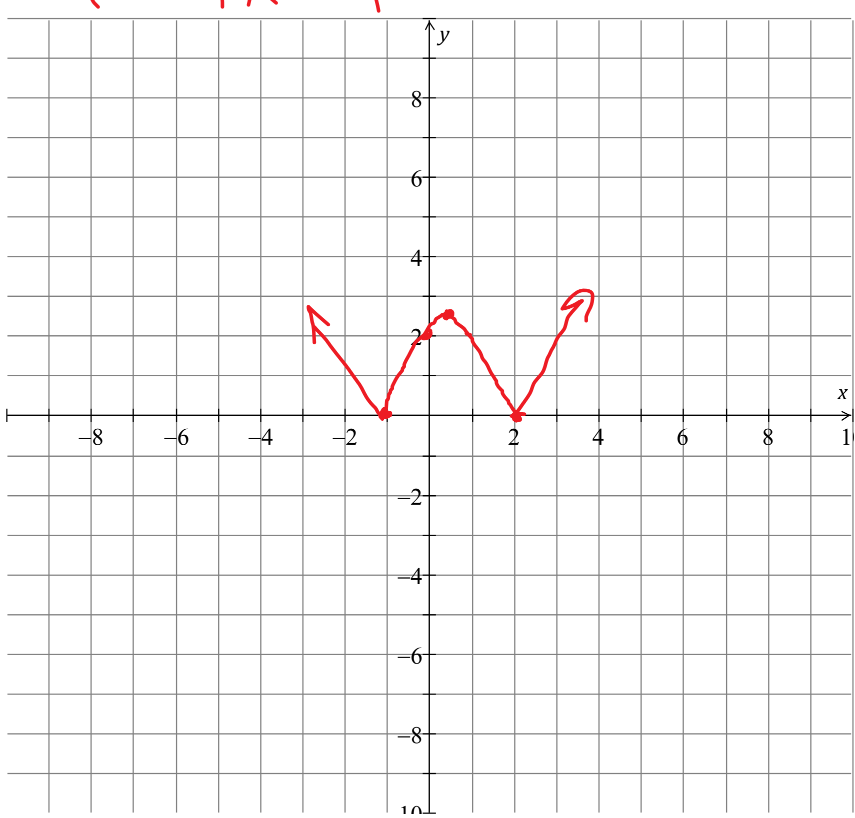
$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1, x = 2$

$(-1,0), (2,0)$

(B) Sketch the graph.



Vertex:

$p = \frac{-b}{2a} = \frac{-(-1)}{2(1)} = 0.5$

$q = |(0.5)^2 - (0.5) - 2|$

$q = |-2.25|$

$q = 2.25$

Vertex: $(0.5, 2.25)$

(C) State the domain and range:

Domain: $\{x \mid x \in R\}$

Range: $\{y \mid y \geq 0, y \in R\}$

(D) Express as a piecewise function:

$$y = \begin{cases} x^2 - x - 2, & x \leq -1, x \geq 2 \\ -(x^2 - x - 2), & -1 < x < 2 \end{cases}$$

Expressing Absolute Value Functions as Piecewise Functions Algebraically
Similarly, we can use a visual representation of the absolute value of a quadratic function to determine the piecewise function. This is an algebraic method using sign diagrams to analyze where the quadratic function is positive or negative.

Example 4:
Express the function $y = |(x-2)^2 - 4|$ as a piecewise function.

$y = (x-2)^2 - 4$

• we need roots

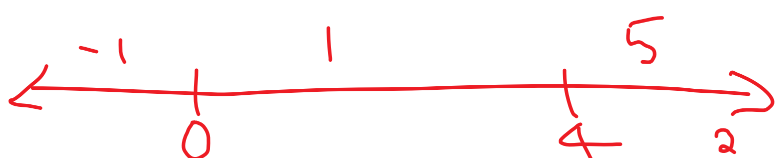
$y = x^2 - 4x + 4 - 4$

$y = x^2 - 4x$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, x = 4$



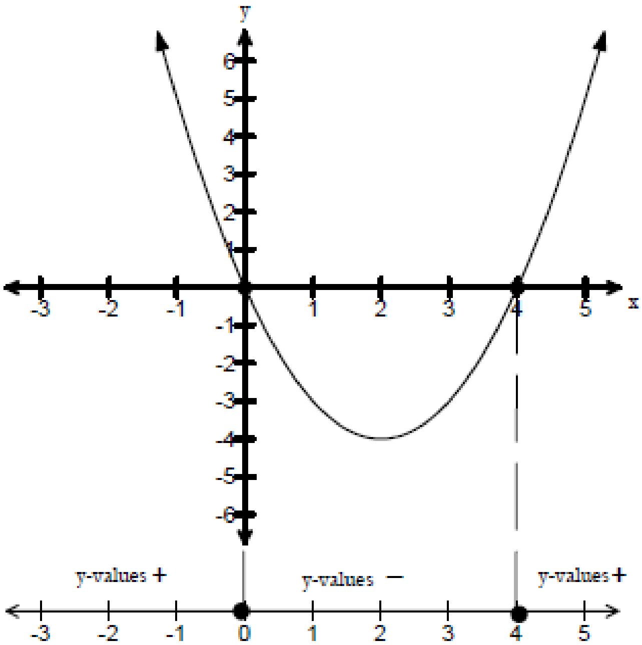
$y = (-1-2)^2 - 4$	$y = (1-2)^2 - 4$	$y = (5-2)^2 - 4$
$y = (-3)^2 - 4$	$y = (-1)^2 - 4$	$y = (3)^2 - 4$
$y = 9 - 4$	$y = 1 - 4$	$y = 9 - 4$
$y = 5$	$y = -3$	$y = 5$
+	—	+

How do we turn negatives to positives?

• multiply by -1.

So where the y-values are negative, we want the negative of the function.

As you can see from the graph below, the test points do in fact show where the graph is below the x-axis. It is this interval where we apply a negative to the function which would invert all y-value from negative to positive.



$$y=\begin{cases}(x-2)^2-4, & x\le 0,\ x\ge 4\\ -[(x-2)^2-4], & 0<x<4\end{cases}$$

Example 5:
Express the function $y = |-3x - 12|$ as a piecewise function.

$-3x-12=0$

$\frac{-3x}{-3}=\frac{12}{-3}$

$x=-4$

$-5 \quad -4 \quad 0$

$y=-3(-5)-12$

$y=15-12$

$y=3$

$+$

$y=-3(0)-12$

$y=-12$

$-$

$$y=\begin{cases}-3x-12, & x\le -4\\ -(-3x-12), & x>-4\end{cases}$$

Example 6:
Express the function $y = |-x^2 + x + 3|$ as a piecewise function.

$-x^2+x+3=0$

$x^2-x-3=0$

$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-3)}}{2(1)}$$

$$x=\frac{1\pm\sqrt{13}}{2}$$

$$x=\frac{1\pm 3.6}{2}$$

$x=2.3,\ x=-1.3$

-2	0	3
$x<-1.3$	-1.3 to 2.3	$x>2.3$
$y=-(-2)^2+(-2)+3$	$y=-(0)^2+0+3$	$y=-(3)^2+3+3$
$y=-3$	$y=3$	$y=-9+6$
		$y=-3$
$-$	$+$	$-$

$$y=\begin{cases}-x^2+x+3, & -1.3\le x\le 2.3\\ -(-x^2+x+3), & x<-1.3,\ x>2.3\end{cases}$$

Textbook Questions: page 375 - 379 #2, 3, 5, 6 a, b, c, 7, 8 a, b, f, 9, 10, 11, 12, 13