Math 2200
7.4A Reciprocal Functions - Linear

Recall from arithmetic, the product of a number and it's reciprocal is always equal 1. For example, $\frac{3}{4}$ is the reciprocal of $\frac{4}{3}$ and $\left(\frac{3}{4}\right) \cdot \left(\frac{4}{3}\right) = \frac{12}{12} = 1$

So for any non-zero, real number a, the reciprocal of a is $\frac{1}{a}$. For a function, $f(x)$, its reciprocal is $\frac{1}{f(x)}$, where $f(x) \neq 0$.

We will be graphing these functions by hand, but I encourage you to use **Desmos graphing** to verify your work and to become more familiar with the concept of reciprocal functions.

Example 1:
What is the reciprocal of $f(x) = 2x + 3$?

$$g(x) = \frac{1}{f(x)} = \frac{1}{2x+3}$$

The Graphs of Linear Functions and their Reciprocals

We will investigate graphing reciprocal functions by looking at the table of values and the graphs of the function $f(x) = x$ and its reciprocal, $(x) = \frac{1}{x}$.

x	$y = x$	$y = \frac{1}{x}$
-10	-10	$-\frac{1}{10}$
-5	-5	$-\frac{1}{5}$
-2	-2	$-\frac{1}{2}$
-1	-1	-1
$-\frac{1}{2}$	$-\frac{1}{2}$	-2
$-\frac{1}{5}$	$-\frac{1}{5}$	-5
$-\frac{1}{10}$	$-\frac{1}{10}$	-10
0	0	undefined
$\frac{1}{10}$	$\frac{1}{10}$	10
$\frac{1}{5}$	$\frac{1}{5}$	5
$\frac{1}{2}$	$\frac{1}{2}$	2
1	1	1
2	2	$\frac{1}{2}$
5	5	$\frac{1}{5}$
10	10	$\frac{1}{10}$

Notice that the function values for $y = \frac{1}{x}$ can be found by taking the reciprocal of the function values for $y = x$.

What is unique about the reciprocals of -1 and 1? Why?

The reciprocal of 1 and -1 is $\frac{1}{1} = 1$ and $\frac{1}{-1} = -1$ respectively.

Why is the reciprocal of 0 undefined?

$\frac{1}{0}$ is undefined

What happens to the value of the reciprocal as the absolute value of a number increases in value?

They trend smaller.

What are Asymptotes?

asymptote: a line whose distance from a given curve approaches zero.

vertical asymptote: for reciprocal functions, occurs at the non-permissible values of the function.
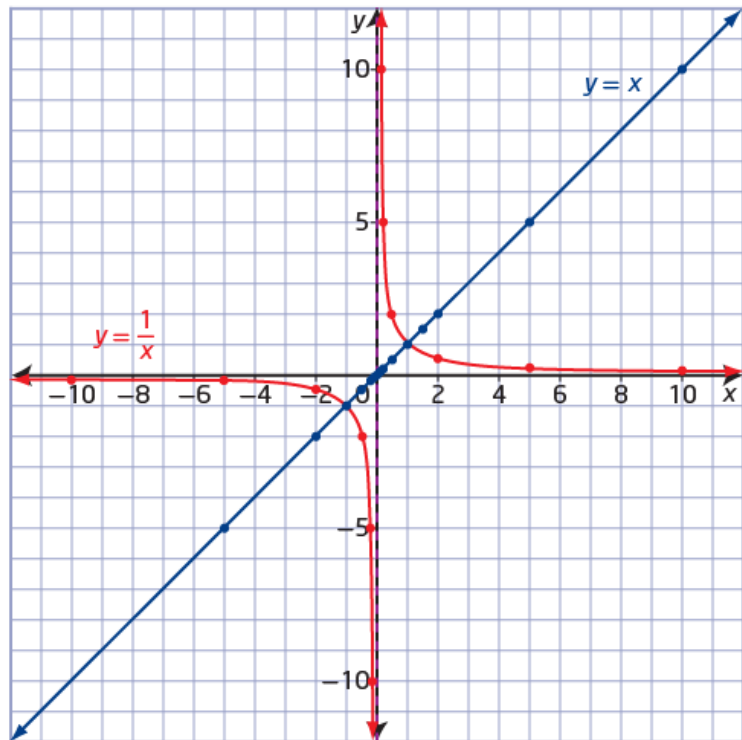
horizontal asymptote: describes the behavior of a graph when $|x|$ is very large.

The horizontal asymptote will be $y = 0$ for all reciprocal graphs discussed in this course.

To find horizontal asymptotes: If the degree, or the largest exponent, of the denominator is bigger than the degree of the numerator, the horizontal asymptote is the x – axis, or the line $y = 0$.

If the degree of the numerator is bigger than the denominator, there is **no** horizontal asymptote.

Graph of $f(x) = x$ and $g(x) = \frac{1}{x}$:

Summary

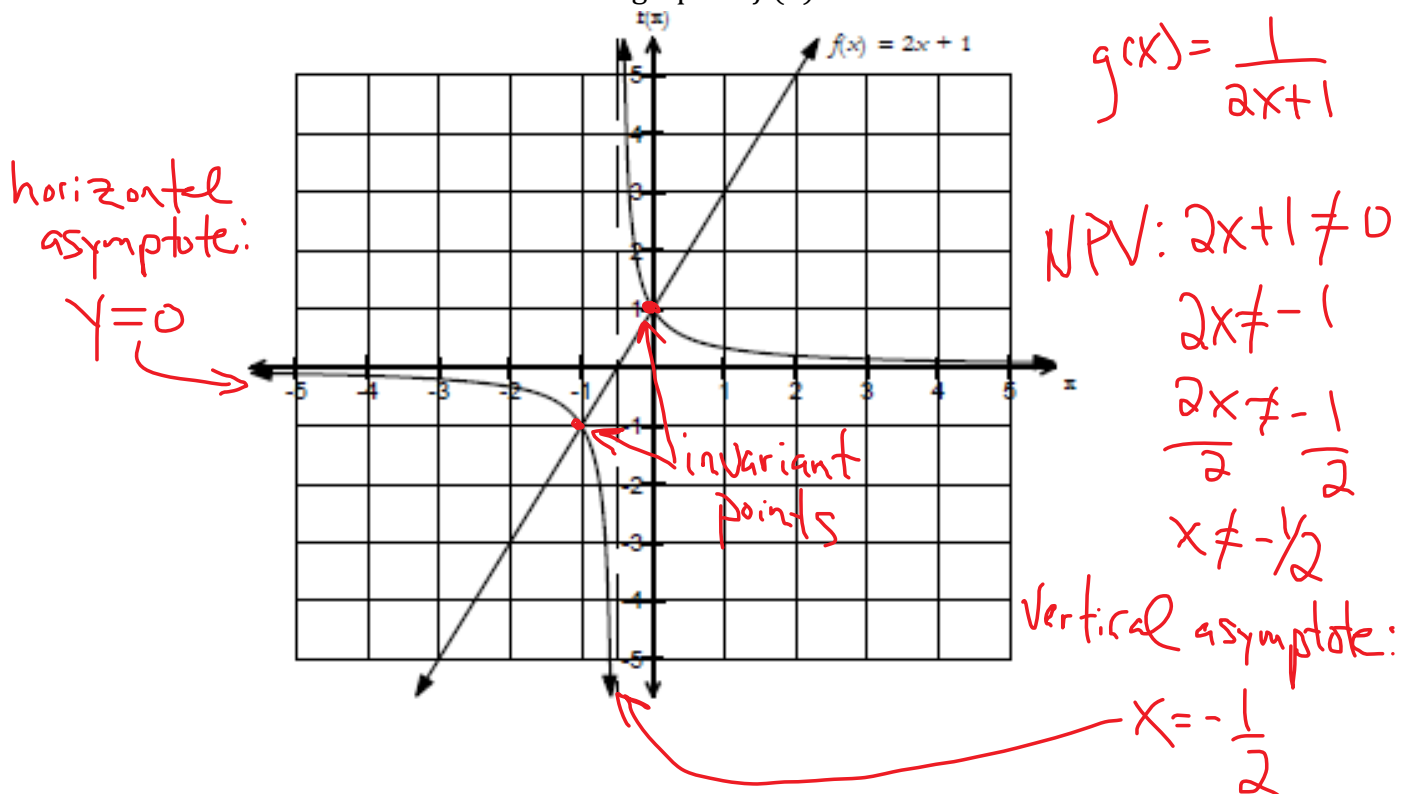| Characteristic	$y = x$	$y = \frac{1}{x}$
Domain	$\{x \mid x \in R\}$	$\{x \mid x \neq 0, x \in R\}$
Range	$\{y \mid y \in R\}$	$\{y \mid y \neq 0, y \in R\}$
End behaviour	• If $x > 0$ and $|x|$ is very large, then $y > 0$ and is very large. • If $x < 0$ and $|x|$ is very large, then $y < 0$ and $|y|$ is very large.	• If $x > 0$ and $|x|$ is very large, then $y > 0$ and is close to 0. • If $x < 0$ and $|x|$ is very large, then $y < 0$ and y is close to 0.
Behaviour at $x = 0$	$y = 0$	undefined, vertical asymptote at $x = 0$
Invariant points		$(-1, -1)$ and $(1, 1)$

Example 2:
When graphing a function and its reciprocal, there are invariant points that remain unchanged. Let's look at the following table to highlight the invariant points using the function $f(x) = 2x + 1$.

x	$f(x)$	$\frac{1}{f(x)}$
1	3	$\frac{1}{3}$
0.5	2	$\frac{1}{2}$
0	1	1
-0.5	0	undefined
-1	-1	-1

invariant points (handwritten annotation)

Notice that the invariant points can be located by setting $f(x) = \pm 1$. These points will help you graph reciprocal functions, as these will be common to both functions and are therefore the points of intersection.

As you explore linear and quadratic reciprocal functions, there are certain characteristics that should be addressed. Let's look at the graph of $f(x) = 2x + 1$.

horizontal asymptote:
$$Y = 0$$

$$g(x) = \frac{1}{2x+1}$$

NPV: $2x + 1 \neq 0$
$$2x \neq -1$$
$$\frac{2x}{2} \neq \frac{-1}{2}$$
$$x \neq -\frac{1}{2}$$

Vertical asymptote:
$$x = -\frac{1}{2}$$

(A) If the point (x, y) is on the graph $y = f(x)$, what do you notice about the point on the graph of the reciprocal function?

The x-coordinate is the same.
The y-coordinate is the reciprocal.

(B) Is the sign of the reciprocal function the same as the sign of the original function?

Yes.

(C) What do you notice about the x-intercept of the linear function, the non-permissible value of the reciprocal function, and the location of the vertical asymptote?

The same.

(D) What is the horizontal asymptote?

y=0 or x-axis

(E) What are the invariant points?

$(0,1)$, $(-1,-1)$

(F) Why do the intervals of decrease on the original function become intervals of increase on the reciprocal function?

As the y-values trend smaller, their reciprocals trend larger.

(G) Why do the intervals of increase on the original function become intervals of decrease on the reciprocal function?

As the y-values trend larger, their reciprocals trend smaller.

(H) What do you notice about the behavior of the reciprocal function as it approaches the asymptotes?

Vertical: as x approaches the asymptote, y approaches -∞ and ∞

horizontal: as x approaches ∞ and -∞, y approaches 0.

Example 3:
Consider the function $f(x) = 2x + 6$

(A) Determine the reciprocal of the function.

$$g(x) = \frac{1}{2x+6}$$

(B) Determine the vertical asymptote of the reciprocal function.

$$2x+6 \neq 0$$
$$2x \neq -6$$

$$x \neq -\frac{6}{2}$$
$$x \neq -3$$

$\therefore$ vertical asymptote: $x = -3$

(C) Determine the invariant points of the function and its reciprocal. $y = \pm 1$

$$2x+6 = -1$$
$$2x = -1-6$$
$$2x = -7$$

$\Rightarrow x = -\frac{7}{2}$
$x = -3.5$
$(-3.5, -1)$

$$2x+6 = 1$$
$$2x = 1-6$$
$$2x = -5$$

$\Rightarrow x = -\frac{5}{2}$
$x = -2.5$
$(-2.5, 1)$

(D) Determine the y-intercept of the reciprocal function

$$x = 0 \quad y = \frac{1}{2(0)+6} = \frac{1}{6}$$

y-intercept: $\frac{1}{6}$

(E) Graph the function and its reciprocal.

$y = 2x+6$

$y = \frac{1}{2x+6}$
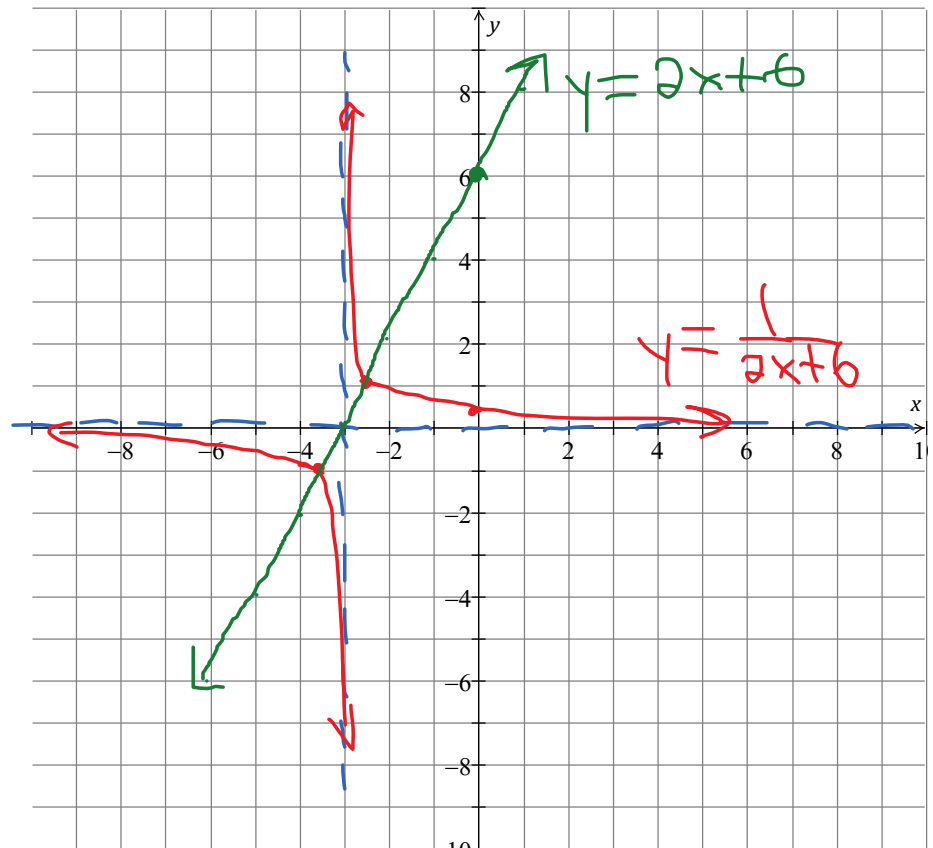
Example 4:
Algebraically determine the **invariant points**, **equations of asymptotes**, and **x- and y-intercepts** for the functions $f(x) = -2x + 6$ and $y = \frac{1}{f(x)}$. Sketch both graphs on the same set of axes.

$$g(x) = \frac{1}{-2x+6}$$

$$-2x + 6 \neq 0$$
$$\frac{-2x}{-2} \neq \frac{-6}{-2}$$
$$x \neq 3$$

VA: $x = 3$

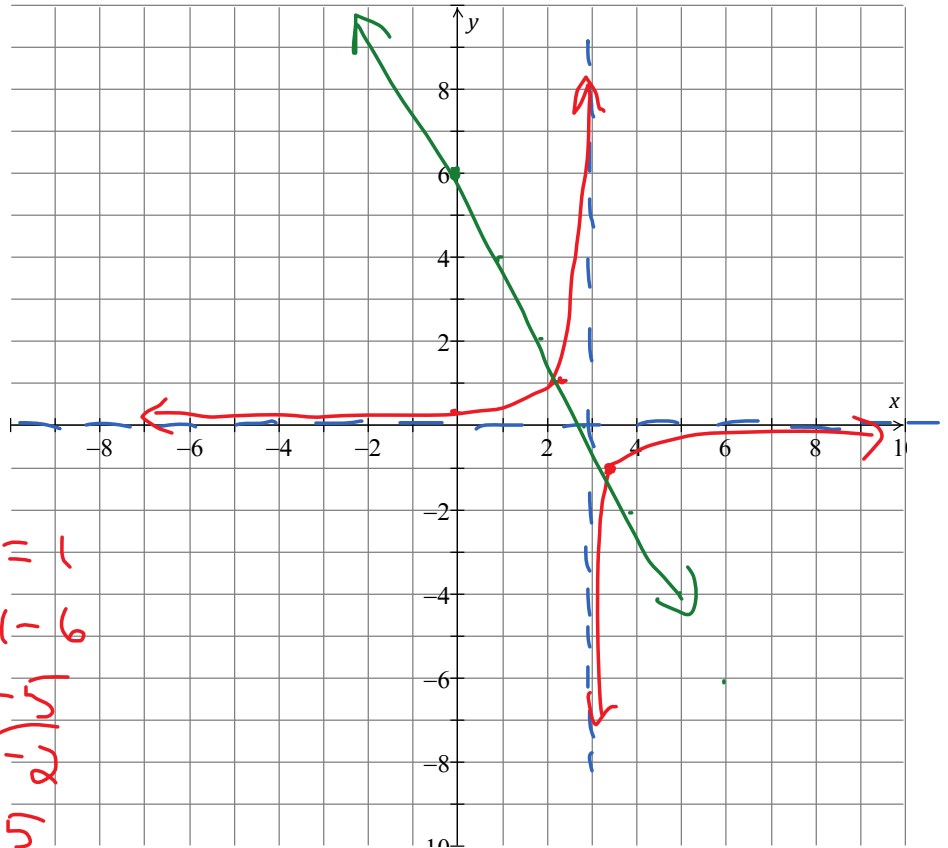|	
$-2x + 6 = -1$	$-2x + 6 = 1$
$-2x = -1 - 6$	$-2x = 1 - 6$
$\frac{-2x}{-2} = \frac{-7}{-2}$	$\frac{-2x}{-2} = \frac{-5}{-2}$
$x = 3.5$	$x = 2.5$
$(3.5, -1)$	$(2.5, 1)$

invariant points

y-int: $x = 0$

$$y = \frac{1}{-2(0)+6} = \frac{1}{6}$$

Example 5:

Consider the function $f(x) = \dfrac{1}{3x-4}$	True	False	Why I (we) think so
(1) There is a vertical asymptote at $x = \frac{4}{3}$	✓		$3x - 4 \neq 0 \Rightarrow x \neq \frac{4}{3}$ $3x \neq 4$
(2) There is a horizontal asymptote at $y = 0$.	✓		Degree of denominator is larger than numerator.
(3) $f(0) = \frac{1}{4}$		✓	$f(0) = \dfrac{1}{3(0)-4} = \dfrac{1}{-4} = -\dfrac{1}{4}$
(4) The point $(4, 8)$ is on the graph of $y = 3x - 4$ and the point $(4, -8)$ is on the graph of $y = \frac{1}{3x-4}$.		✓	The reciprocal of a function would not produce a negative range value. It would reciprocate the range value.

Textbook Questions: page 403 - 405 #1a,b, 2a,b, 3a,b, 5a,b, 6a, 7, 10a, 17