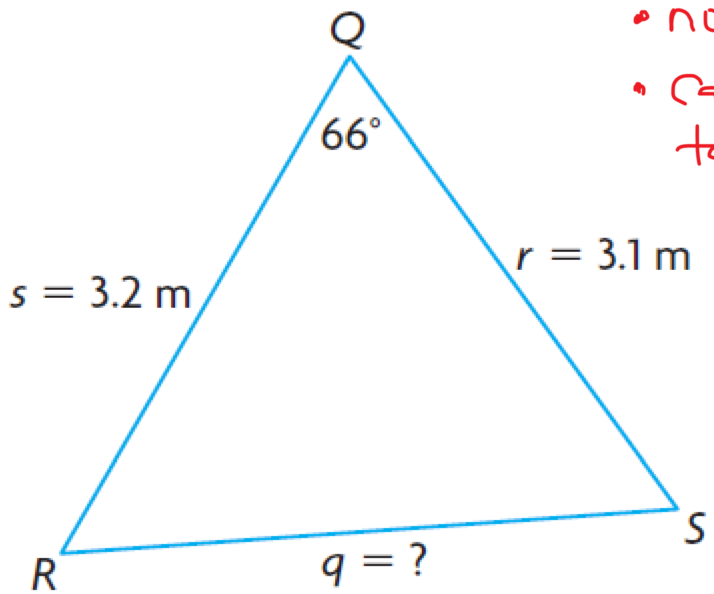
Math 2201 **Date:**____________________

3.3 Proving and Applying the Cosine Law

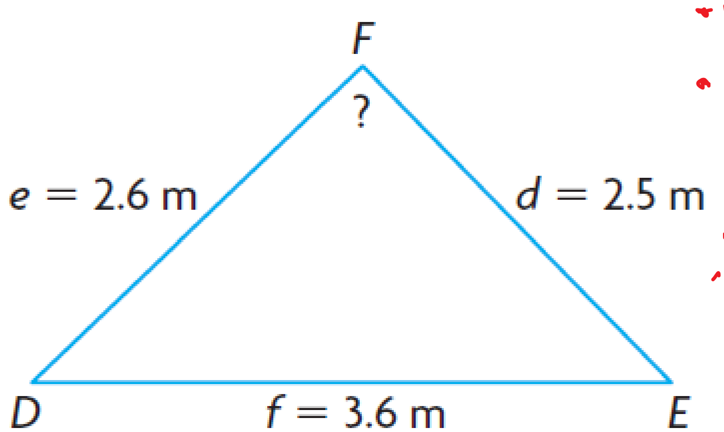
The Sine Law may not be able to help us solve for missing sides and/or angles in a triangle. Recall that the **Sine Law** can be used when we are dealing with **two sides and two angles, one of which is unknown**. In other words you need a complete ratio of side and the angle plus the side or angle opposite the side or angle you are trying to find.

Sometimes we may be given a situation in which we are dealing with three sides and one angle - one of which is unknown.



- no complete ratio
- can't find an angle to make a complete ratio.

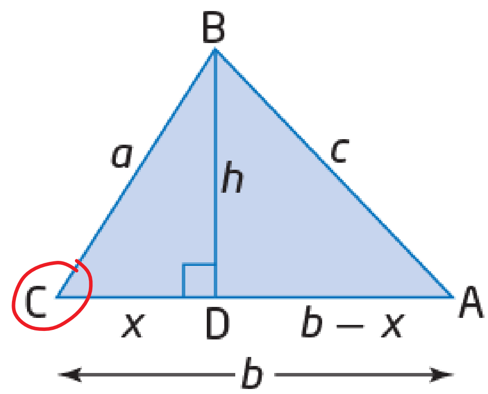
∴ Can NOT use the Sine Law.



- no complete ratio
- no way to find any angle.

∴ Can NOT use the Sine Law.

The cosine law describes the relationship between the cosine of an angle and the lengths of the three sides of any angle. Lets take a look at how this formula was developed:



ΔBCD

$\cos C = \frac{x}{a}$

$x = a\cos C$ $\qquad a^2 = h^2 + x^2$

ΔABD

$c^2 = h^2 + (b-x)^2$ $\qquad (b-x)(b-x)$

$\qquad b^2 - 2bx + x^2$

$c^2 = h^2 + b^2 - 2bx + x^2$

$c^2 = h^2 + x^2 + b^2 - 2bx$

$c^2 = a^2 + b^2 - 2ba\cos C$

The Cosine Law

For any triangle, ΔABC, where a, b, and c are the lengths of the sides opposite ∠A, ∠B and ∠C, respectively. Then

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$ ⟵ provided for tests and exams.

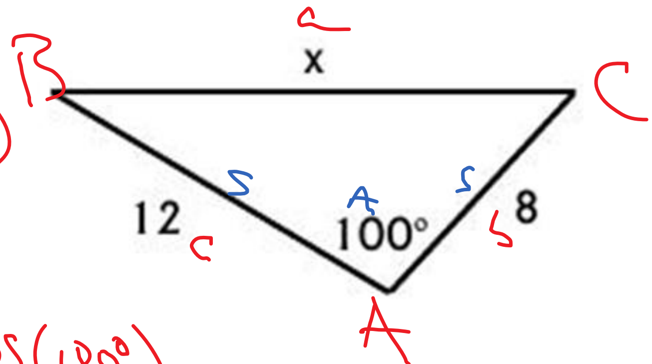
$$b^2 = a^2 + c^2 - 2ac \cdot \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

Example 1

Determine the measure of x.

- Look for side-angle-side (SAS)



$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = (8)^2 + (12)^2 - 2(8)(12)\cos(100°)$

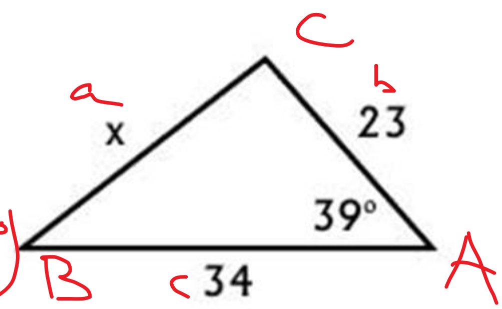
$a^2 = 241.3404$

$\sqrt{a^2} = \sqrt{241.3404}$

$a = 16$ units

Example 2

Determine the measure of x.



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a^2 = (23)^2 + (34)^2 - 2(23)(34)\cos(39^\circ)$$

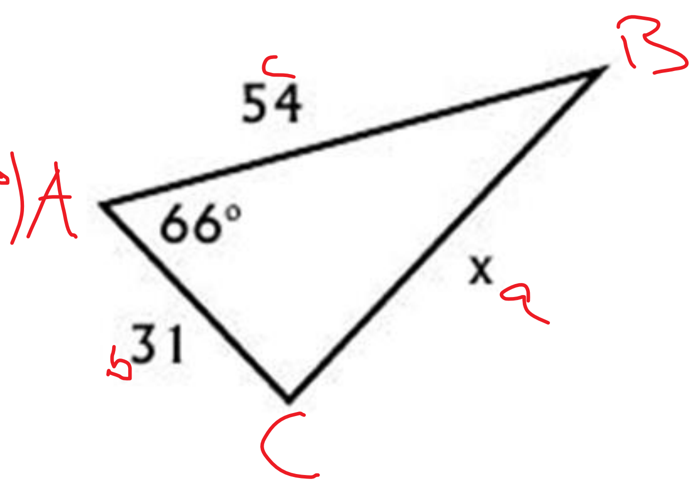
$$a^2 = 469.5437$$

$$\sqrt{a^2} = \sqrt{469.5437}$$

$$a = 22 \text{ units}$$

Example 3

Determine the measure of x.



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a^2 = (31)^2 + (54)^2 - 2(31)(54)\cos(66^\circ)$$

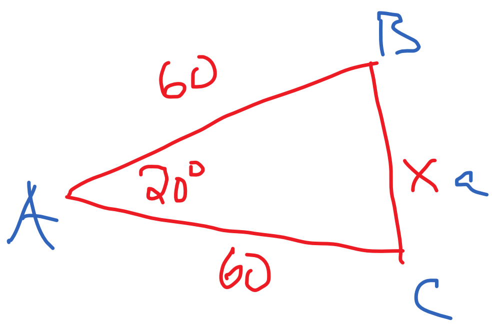
$$a^2 = 2515.2457$$

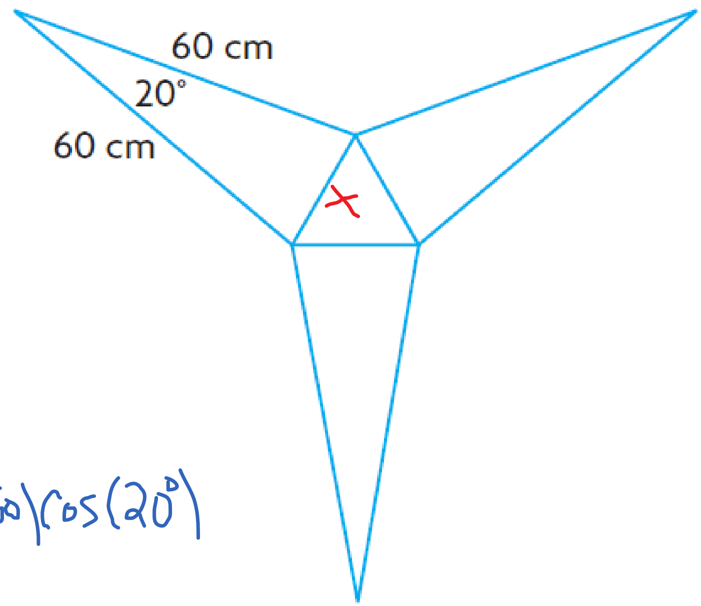
$$\sqrt{a^2} = \sqrt{2515.2457}$$

$$a = 50 \text{ units}$$

Example 4

A three-pointed star is made up of an equilateral triangle and three congruent isosceles triangles. Determine the length of each side of the equilateral triangle in this three-pointed star. Round the length to the nearest centimetre.





$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

$$a^2 = (60)^2 + (60)^2 - 2(60)(60)\cos(20°)$$

$$a^2 = 434.213$$

$$\sqrt{a^2} = \sqrt{434.213}$$

$$a = 21 \text{ cm}$$

Each side of the equilateral triangle is 21cm.

We can also use the Cosine Law to solve for missing angles. We can use the formula as is, but it's better to rearrange the formula and isolate the angle. Solve for cosA:

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

$$b^2 + c^2 - 2bc \cdot \cos A = a^2$$

$$\frac{\cancel{-2bc} \cdot \cos A}{\cancel{-2bc}} = \frac{a^2 - b^2 - c^2}{-2bc}$$

$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$$

$$\cos A = -\frac{(a^2 - b^2 - c^2)}{2bc}$$

$$\cos A = \frac{-a^2 + b^2 + c^2}{2bc}$$

$$\boxed{\cos A = \frac{b^2 + c^2 - a^2}{2bc}}$$

For any triangle, ΔABC, where a, b, and c are the lengths of the sides opposite ∠A, ∠B and ∠C, respectively. Then

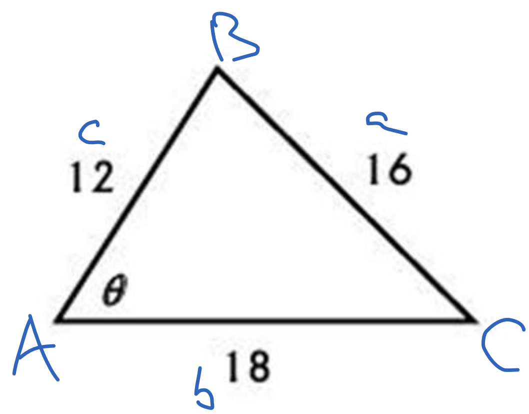
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 5

Determine the value of θ to the nearest degree.



- Look for is side-side-side (SSS)

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\theta = \frac{(18)^2 + (12)^2 - (16)^2}{2(18)(12)}$$

$$\cos\theta = \frac{212}{432}$$

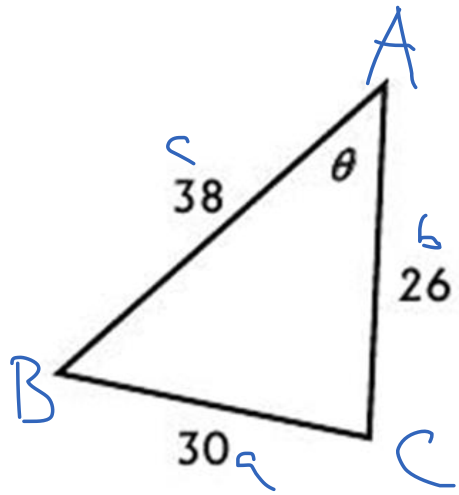
$$\cos\theta = 0.4907$$

$$\theta = \cos^{-1}(0.4907)$$

$$\theta = 61^\circ$$

Example 6

Determine the value of θ to the nearest degree.



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\theta = \frac{(26)^2 + (38)^2 - (30)^2}{2(26)(38)}$$

$$\cos\theta = \frac{1220}{1976}$$

$$\cos\theta = 0.6174$$

$$\theta = \cos^{-1}(0.6174)$$

$$\theta = 52^\circ$$

Common Mistakes

When working with the cosine law, students sometimes incorrectly apply the order of operations. When asked to simplify $a^2 = 365 - 360\cos 70^\circ$, for example, they often write $a^2 = 5\cos 70^\circ$. To avoid this error, teachers should emphasize that multiplication is to be completed before subtraction.

Remember
Brackets
Exponents
Divide
Multiple
Add
Subtract

$$a^2 = 365 - 360\cos 70^\circ$$

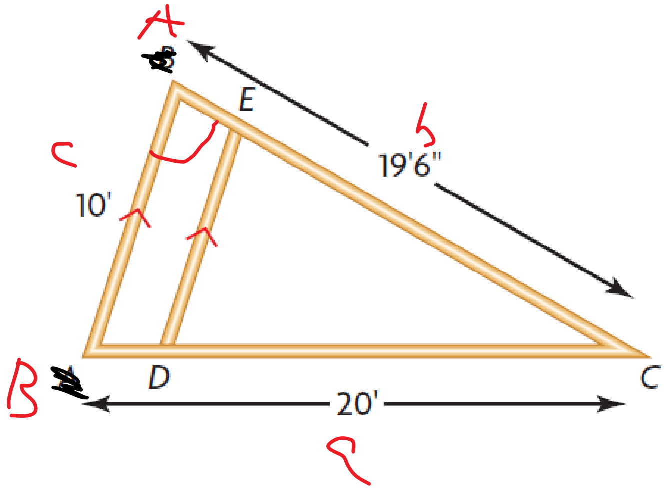
$$a^2 = 365 - 123.1273$$

$$\sqrt{a^2} = \sqrt{241.8727}$$

$$a = 16 \text{ units}$$

Example 7

The diagram at the right shows the plan for a roof, with support beam $\overline{DE}$ parallel to $\overline{AB}$. The local building code requires the angle formed at the peak of a roof to fall within a range of 70° to 80° so that snow and ice will not build up. Will this plan pass the local building code?



$19'6'' = 19.5$ ft.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{(19.5)^2 + (10)^2 - (20)^2}{2(19.5)(10)}$$

$$\cos A = \frac{80.25}{390}$$

$\cos A = 0.2058$

$A = \cos^{-1}(0.2058)$

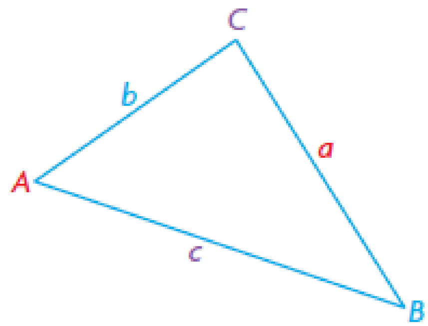
$A = 78°$

Yes, this plan will pass the local building code.

In Summary

Key Idea

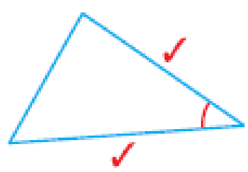
- The cosine law can be used to determine an unknown side length or angle measure in an acute triangle.

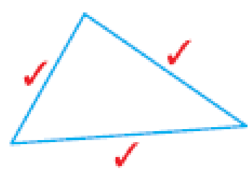


$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Need to Know

- You can use the cosine law to solve a problem that can be modelled by an acute triangle when you know:
  - two sides and the contained angle.
  - all three sides.





- The contained angle is the angle between two known sides.
- When using the cosine law to determine an angle, you can:
  - substitute the known values first, then solve for the unknown angle.
  - rearrange the formula to solve for the cosine of the unknown angle, then substitute and evaluate.

Textbook Questions: page: 150 - 153; # 1, 2, 3, 4, 5, 6, 7, 9,13, 15