Math 2201 **Date:**________________

7.3 Solving Quadratic Equations Using the Quadratic Formula

The Quadratic Formula is derived from a method of solving quadratic equations called completing the square. The Quadratic Formula works for all quadratic equations, but more importantly, it works for quadratic equations that are not factorable using product/sum or decomposition.

Given a quadratic equation in standard form, $ax^2 + bx + c = 0$, the **Quadratic Formula** is defined as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1:
Find the roots of $y = \overset{a}{x^2} + \overset{b}{10}x + \overset{c}{21}$ using the quadratic formula.

$x^2 + 10x + 21 = 0$
$(x+3)(x+7) = 0$
$x = -3, x = -7$

$x = \frac{-10 \pm \sqrt{10^2 - 4(1)(21)}}{2(1)}$

$x = \frac{-10 \pm \sqrt{100 - 84}}{2}$

$x = \frac{-10 \pm \sqrt{16}}{2}$

$x = \frac{-10 \pm 4}{2}$

$x = \frac{-10+4}{2}, x = \frac{-10-4}{2}$

$x = -\frac{6}{2}, x = -\frac{14}{2}$

$x = -3, x = -7$

$(-3, 0), (-7, 0)$

Example 2:
Solve using the quadratic formula, express your answer as an exact value.

$$\overset{a}{5x^2} - \overset{b}{7x} - \overset{c}{1} = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(5)(-1)}}{2(5)}$$

$$x = \frac{7 \pm \sqrt{69}}{10}$$

$$x = \frac{7 + \sqrt{69}}{10}, \quad x = \frac{7 - \sqrt{69}}{10}$$

$$\left(\frac{7 + \sqrt{69}}{10}, 0\right), \left(\frac{7 - \sqrt{69}}{10}, 0\right)$$

Example 3:
Solve using the quadratic formula, express your answer as an exact value.

$$6x^2 - 3 = 7x$$

$$6x^2 - 7x - 3 = 0$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(6)(-3)}}{2(6)}$$

$$x = \frac{7 \pm \sqrt{49 + 72}}{12}$$

$$x = \frac{7 \pm \sqrt{121}}{12}$$

$$x = \frac{7 \pm 11}{12}$$

$$x = \frac{7+11}{12}, \quad x = \frac{7-11}{12}$$

$$x = \frac{18}{12}, \quad x = \frac{-4}{12}$$

$$x = \frac{3}{2}, \quad x = -\frac{1}{3}$$

$$\left(\frac{3}{2}, 0\right), \left(-\frac{1}{3}, 0\right)$$

Example 4:
Solve using the quadratic formula, express your answer to the nearest hundredth.

$$x(3x + 5) = 2$$

$$3x^2 + 5x - 2 = 0$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(3)(-2)}}{2(3)}$$

$$x = \frac{-5 \pm \sqrt{49}}{6}$$

$$x = \frac{-5 \pm 7}{6}$$

$$x = \frac{-5 + 7}{6}, \quad x = \frac{-5 - 7}{6}$$

$$x = \frac{2}{6}, \quad x = \frac{-12}{6}$$

$$x = 0.33 \qquad x = -2$$

$$(0.33, 0) \qquad (-2, 0)$$

Example 5:
Solve using the quadratic formula, express your answer to the nearest hundredth.

$$-4 + 3x^2 = x$$

$3x^2 - x - 4 = 0$

$x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-4)}}{2(3)}$

$x = \dfrac{1 \pm \sqrt{49}}{6}$

$x = \dfrac{1 \pm 7}{6}$

$x = \dfrac{1+7}{6}, \quad x = \dfrac{1-7}{6}$

$x = \dfrac{8}{6}, \quad x = -\dfrac{6}{6}$

$x = 1.33, \quad x = -1$

$(1.33, 0), (-1, 0)$

Negative Radicands

When the radicand in the quadratic formula simplifies to a negative number we are able to say that the corresponding quadratic equation has no solution, or in other words, no real roots/zeros/x-intercepts. It is not the intention of this course to introduce the imaginary number system but you will need to be aware that there is no real solution to the square root of a negative number.

Example 6:

Solve using the quadratic formula, express your answer as an exact value.

$$16x^2 + 8x = -5$$

$16x^2 + 8x + 5 = 0$

$x = \dfrac{-8 \pm \sqrt{8^2 - 4(16)(5)}}{2(16)}$

$x = \dfrac{-8 \pm \sqrt{64 - 320}}{32}$

$\rightarrow x = \dfrac{-8 \pm \sqrt{-256}}{32}$

No real roots.

Example 7:

Solve using the quadratic formula, express your answer as an exact value.

$$2x - 3 = 14x^2$$

$0 = \overset{a}{14}x^2 \overset{b}{-2}x + \overset{c}{3}$

$x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(14)(3)}}{2(14)}$

$x = \dfrac{2 \pm \sqrt{4 - 168}}{28}$

$x = \dfrac{2 \pm \sqrt{-164}}{28}$

No real roots.

Common Errors

Some common errors occur when simplifying the quadratic formula. Students may:

- Apply the quadratic formula without ensuring the equation is written in standard form.
- Use an incorrect version such as $x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$ of the rather than the correct form of the quadratic formula.
- Incorrectly substitute the value for b and/or c back into the equation. When solving $x^2 - 5 = 0$, for example, students often substitute $b = 1$ rather than $b = 0$.
- Incorrectly produce two possible common errors if the b-value is negative.
  - i. If $b = -2$ then $-b = -(-2) = -2$
  - ii. If $b = -2$, then $b^2 = -2^2 = -4$
- Incorrectly simplify when applying the quadratic formula.
  - i. $\frac{2 \pm 4\sqrt{5}}{2} = \pm 2\sqrt{5}$
  - ii. $\frac{8 \pm \sqrt{5}}{2} = 4 \pm \sqrt{5}$
- Do not recognize that the $\pm$ results in two solutions. Be sure to work through the solutions separately, showing calculations for both the positive solution and the negative solution.

Example 6:

Mary, David, and Ron are students in a group. They are given the equation $A = x^2 + 3x - 110$ where A represents the area of a field and x represents the width in metres. The students were asked to find the width if the area was 100 m^2. Each student decided to solve the equation using their own preferred method. Here are their solutions:

Mary

$x^2 + 3x - 110 = 100$

$x^2 + 3x - 210 = 0$

$x = \frac{-3 \pm \sqrt{9-(4)(1)(-210)}}{2}$

$x = \frac{3 \pm \sqrt{-831}}{2}$

$x = \frac{3 \pm 28.827}{2}$

$x = 15.9$ or $x = -12.9$

width is 15.9 m

David

$x^2 + 3x - 110 = 100$

$x^2 + 3x - 10 = 0$ -210

$(x+5)(x-3) = 0$

$x = -5$ or $x = 3$

width is 3 m

Ron

$x^2 + 3x - 110 = 0$ 100

$x = \frac{-3 \pm \sqrt{9-(4)(1)(-110)}}{2}$

$x = \frac{-3 \pm \sqrt{9+440}}{2}$

$x = \frac{-3 \pm \sqrt{449}}{2}$

$x = \frac{-3 \pm 21.2}{2}$

$x = 9.1$ or $x = -12.1$

width is 9.1 m

Identify and explain the errors in the students' work, then write the correct solution.

$x^2+3x-210=0$

$x=\dfrac{-3\pm\sqrt{(3)^2-4(1)(-210)}}{2(1)}$

$x=\dfrac{-3\pm\sqrt{9+840}}{2}$

$x=\dfrac{-3\pm\sqrt{849}}{2} \rightarrow x=\dfrac{-3\pm 29.1}{2}$

$x=\dfrac{-3+29.1}{2}, x=\dfrac{-3-29.1}{2}$

$x=13.05, x=-16.1$

Exact vs Approximate Solutions

It is important to distinguish between exact and approximate solutions. Consider the following equations.

Example 8:
Solve each of the equations using the quadratic formula.

A. $x^2+2x-8=0$

$x=\dfrac{-2\pm\sqrt{(2)^2-4(1)(-8)}}{2}$

$x=\dfrac{-2\pm\sqrt{36}}{2}$

$x=\dfrac{-2\pm 6}{2}$

$x=\dfrac{-2+6}{2}, x=\dfrac{-2-6}{2}$

$x=\dfrac{4}{2}, x=-\dfrac{8}{2}$

$x=2, x=-4$

$(2,0), (-4,0)$

B. $x^2+x-4=0$

$x=\dfrac{-1\pm\sqrt{1^2-4(1)(-4)}}{2(1)}$

$x=\dfrac{-1\pm\sqrt{17}}{2}$

$x=\dfrac{-1+\sqrt{17}}{2}, x=\dfrac{-1-\sqrt{17}}{2}$

$\left(\dfrac{-1+\sqrt{17}}{2}, 0\right), \left(\dfrac{-1-\sqrt{17}}{2}, 0\right)$

(A) What is the value under the square root in equation A? Is it a perfect square? How many roots exist and are they exact or approximate?

36. Yes, 36 is a perfect square. There are two roots. The roots are exact.

(B) What is the value under the square root in equation B? Is it a perfect square? How many roots exist and are they exact or approximate?

17. No, 17 is not a perfect square. There are two roots. The roots are approximate.

(C) What values of $b^2 - 4ac$ could lead to approximate answers? What values of $b^2 - 4ac$ could lead to exact answers?

If b^2-4ac is not a perfect square, we get approximate answers.

If b^2-4ac is a perfect square, we get exact answers.

(D) Which equation can also be solved by factoring?

A: x^2+2x-8
$(x+4)(x-2)$

Equation can be solved by factoring.

(E) What connection can be made between an equation that is factorable and the value of $b^2 - 4ac$?

If b^2-4ac is a perfect square, the equation is factorable.

Example 9:
Solve using the quadratic formula, express your answer as an approximation or decimal.

$$2x^2 + 8x - 5 = 0$$

$$x = \frac{-8 \pm \sqrt{8^2-4(2)(-5)}}{2(2)}$$

$$x = \frac{-8 \pm \sqrt{104}}{4}$$

$$x = \frac{-8 \pm 10.2}{4}$$

$$x = \frac{-8+10.2}{4}, \quad x = \frac{-8-10.2}{4}$$

$$x = 0.6, \quad x = -9.1$$

$(0.6, 0)$, $(-9.1, 0)$

Example 10:
Solve using the quadratic formula, express your answer as an approximation or decimal.

$$5x^2 - 10x + 3 = 0$$

$$X = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(5)(3)}}{2(5)}$$

$$X = \frac{10 \pm \sqrt{40}}{10}$$

$$X = \frac{10 \pm 6.3}{10}$$

$$X = \frac{10 + 6.3}{10}, \quad X = \frac{10 - 6.3}{10}$$

$$X = 1.6, \quad X = 0.4$$

$(1.6, 0), (0.4, 0)$

Quadratic Equations Resulting in Single Roots

Example 11:

(A) Solve using the quadratic formula:

$$4x^2 - 16x + 16 = 0$$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(4)(16)}}{2(4)}$$

$$x = \frac{16 \pm \sqrt{256 - 256}}{8}$$

$$x = \frac{16 \pm 0}{8}$$

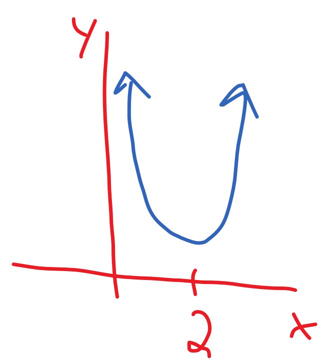
$$x = \frac{16}{8}$$

$$x = 2$$

$$(2, 0)$$

(B) Describe the conditions for a, b, and c in $ax^2 + bx + c = 0$ that are necessary for the quadratic formula to result in only one possible root.

$$b^2 - 4ac = 0$$



Exact Solutions Involving Radicals
We often want our answers to be exact, reduced radical form. Recall from earlier in this course how we reduce radicals.

Example 12:
Reduce each radical:

$\sqrt{8}$ $\sqrt{12}$ $\sqrt{18}$ $\sqrt{32}$ $\sqrt{72}$ $\sqrt{128}$ $\sqrt{432}$

$=\sqrt{4}\sqrt{2}$ $=\sqrt{4}\sqrt{3}$ $=\sqrt{9}\sqrt{2}$ $=\sqrt{16}\sqrt{2}$ $=\sqrt{36}\sqrt{2}$ $=\sqrt{64}\sqrt{2}$ $=\sqrt{144}\sqrt{3}$

$=2\sqrt{2}$ $=2\sqrt{3}$ $=3\sqrt{2}$ $=4\sqrt{2}$ $=6\sqrt{2}$ $=8\sqrt{2}$ $=12\sqrt{3}$

Example 13:
Solve using the quadratic formula, express your answer in reduced radical form.

$$2x^2 + 8x - 5 = 0$$

$$x = \frac{-8 \pm \sqrt{(8)^2 - 4(2)(-5)}}{2(2)}$$

$$x = \frac{-8 \pm \sqrt{64 + 40}}{4}$$

$$x = \frac{-8 \pm \sqrt{104}}{4}$$

$$x = \frac{-8 \pm \sqrt{4}\sqrt{26}}{4}$$

$$x = \frac{-8 \pm 2\sqrt{26}}{4}$$

$$x = \frac{-4 \pm \sqrt{26}}{2}$$

$$x = \frac{-4 + \sqrt{26}}{2}, \quad x = \frac{-4 - \sqrt{26}}{2}$$

$$\left(\frac{-4 + \sqrt{26}}{2}, 0\right), \left(\frac{-4 - \sqrt{26}}{2}, 0\right)$$

Example 14:
Solve using the quadratic formula, express your answer in reduced radical form.

$$5x^2 - 10x + 3 = 0$$

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(5)(3)}}{2(5)}$$

$$x = \frac{10 \pm \sqrt{100 - 60}}{10}$$

$$x = \frac{10 \pm \sqrt{40}}{10}$$

$$x = \frac{10 \pm \sqrt{4}\sqrt{10}}{10}$$

$$x = \frac{10 \pm 2\sqrt{10}}{10}$$

$$x = \frac{5 \pm \sqrt{10}}{5}$$

$$x = \frac{5 + \sqrt{10}}{5}, \quad x = \frac{5 - \sqrt{10}}{5}$$

$$\left(\frac{5 + \sqrt{10}}{5}, 0\right), \left(\frac{5 - \sqrt{10}}{5}, 0\right)$$

Textbook Questions: page 419 - 421 #1, 2, 3, 4, 5, 6, 7, 9, 10