Math 3201

2.6A Introduction to Combinations

Permutations vs. Combinations

Permutation: an arrangement of objects in which order matters

Combination: is a grouping of objects where order does not matter

Example1:
Identify each of the following as a permutation or a combination problem.

(A) "My fruit salad contains apples, grapes and strawberries." Whether it is a combination of "strawberries, grapes and apples" or "grapes, apples and strawberries", it is the same fruit salad.

Order does not matter. Combination.

(B) "The passcode to my iPhone is 1786". The order 6871 or 7618 would not work. It has to be exactly 1786.

Order does matter. Permutation.

Example 2:
An assignment consists of three questions, A, B, C, and students are required to attempt two.

(A) If order matters, determine the number of permutations.

AB AC BC
BA CA CB

(B) If order does NOT matter, determine the number of combinations.

AB = BA   AC = CA   BC = CB
1   2   3

C = 3   Combinations will always be fewer than permutations.

Example 3:
In a lottery, six numbers from 1 to 49 are selected, hence Lotto 6/49. A winning ticket must contain those same six numbers.

(A) If order matters, determine the number of permutations.

$n = 49$ $\quad {}_{49}P_6 = \underline{49} \cdot \underline{48} \cdot \underline{47}\ \underline{46} \cdot \underline{45} \cdot \underline{44}$

$r = 6$ $\quad = 10{,}068{,}347{,}500$

(B) If order does NOT matter. In other words, as long as you have the same six numbers, but they can be arranged in any order. Determine the number of combinations.

$$C = \frac{10\ 068\ 347\ 500}{6!} = 13{,}987{,}816$$

Summary
The number of combinations will be **less** than the number of permutations since combinations eliminate the effect of groupings containing the same elements arranged in a different order. For example, AB and BA would count as two permutations but only as one combination.

We calculate the number of combinations by dividing the number of permutations by the number of ways we can arrange the elements that are being used or $r!$. This will eliminate the effect of groupings that have the exact same elements, just written in a different order.

Formula for Combinations

$$ {}_nC_r = \frac{{}_nP_r}{r!} $$

Since ${}_nP_r = \frac{n!}{(n-r)!}$,

$$ {}_nC_r = \frac{\frac{n!}{(n-r)!}}{r!} $$

$$_nC_r = \frac{n!}{(n-r)!} \cdot \frac{1}{r!}$$

$$\boxed{_nC_r = \frac{n!}{r!\,(n-r)!}}$$

Different Notation:

$_nC_r$, can also be represented by the notation $\binom{n}{r}$ and for both we say, "n choose r".

Keyword

Example 4:

Determine the number of different three person teams that can be formed a total of nine volunteers.

$n=9$

$r=3$

$$_9C_3 = \frac{9!}{3!\,(9-3)!} = \frac{9!}{3!\,6!} = 84$$

or

$$P = \underline{9} \times \underline{8} \times \underline{7} = 504$$

$$C = \frac{504}{3!} = 84$$

Example 5:

A restaurant serves 10 flavours of ice cream. Danielle has ordered a large sundae with three scoops. How many ice cream combinations does Danielle have to choose from, if she wants a each scoop to be a different flavour.

$n=10$

$r=3$

$$_{10}C_3 = \frac{10!}{3!\,(10-3)!} = \frac{10!}{3!\,7!} = \boxed{120}$$

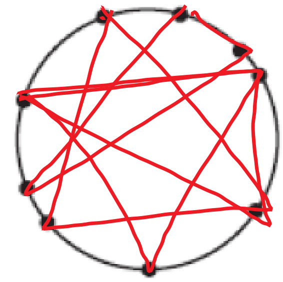
or

$$P = \underline{10} \times \underline{9} \times \underline{8} = 720$$

$$C = \frac{720}{3!} = \boxed{120}$$

Example 6:
There are 9 dots randomly placed on a circle. How many triangles can be formed within the circle?

$$C = \frac{9 \times 8 \times 7}{3!} = 84$$



Example 7:
Tanya is the coach of a Pole Push team that consists of nine players: 5 male and 4 female. In each competition, teams of four complete against each other to push competitors out of a circle. The team that is successful wins.

(A) How many different four person teams does Tanya have to choose from for an all male competition?

$n = 5$
$r = 4$

$$\binom{5}{4} = {}_5C_4 = \frac{5!}{4!(5-4)!} = \frac{5!}{4!\,1!} = 5$$

(B) How many different four person teams does Tanya have to choose from, with two males and two females, for a mixed competition?

2 males: ${}_5C_2$ 2 females: ${}_4C_2$

$$C_T = {}_5C_2 \times {}_4C_2$$

$$= \frac{5!}{2!(5-2)!} \times \frac{4!}{2!(4-2)!}$$

$$= \frac{5!}{2!\,3!} \times \frac{4!}{2!\,2!}$$

$$= 10 \times 6$$

$$= 60$$

Combination Problems Involving Conditions/Cases

Example 7:
A planning committee is to be formed for a school-wide Earth Day program. There are 13 volunteers: 8 teachers and 5 students. In how many different ways can the principal choose a 4 – person committee that has **at least** 1 teacher?

at least 1: 1 or 2 or 3 or 4

1 teacher and 3 students: $_8C_1 \times {_5C_3} = 8 \times 10 = 80$

or

2 teachers and 2 students: $_8C_2 \times {_5C_2} = 28 \times 10 = 280$

or

3 teachers and 1 student: $_8C_3 \times {_5C_1} = 56 \times 5 = 280$

or

4 teachers and 0 students: $_8C_4 \times {_5C_0} = 70 \times 1 = 70$

$C_T = 710$

Example 8:
The student council decides to form a subcommittee of 5 members to plan their Christmas Concert. There are a total of 11 student members: 5 males and 6 females.

(A) Determine how many different ways the subcommittee can consist of exactly three females.

combination

3 females, 2 males

$$C_T = C_F \times C_M$$

$$C_T = {_6C_3} \times {_5C_2}$$

$$C_T = 20 \times 10 = 200$$

(B) Determine how many different ways the subcommittee can consist of at least three females.

at least 3 females: 3 or 4 or 5

3f and 2m $_6C_3 \times {_5C_2} = 20 \times 10 = 200$

or +

4f and 1m $_6C_4 \times {_5C_1} = 15 \times 5 = 75$

or +

5F and 0m $_6C_5 \times {_5C_0} = 6 \times 1 = 6$

$C = 281$

(C) Determine how many different ways the subcommittee can consist of at least one female.

We'll use information obtained in (B).

1F and 4m $= {_6C_1} \times {_5C_4} = 6 \times 5 = 30$

or +

2f and 3m $= {_6C_2} \times {_5C_3} = 15 \times 10 = 150$

180

+ 281 (from B)

461

Your turn:

1. From a standard deck of 52 cards, how many 5-card hands have:

(A) at least three red cards 3 or 4 or 5

3 red and 2 black : $_{26}C_3 \times {}_{26}C_2 = 2600 \times 325 = 845000$
or
4 red and 1 black : $_{26}C_4 \times {}_{26}C_1 = 14950 \times 26 = 388700$
5 red and 0 black $_{26}C_5 \times {}_{26}C_0 = 65780 \times 1 = 65780$

$$845000 + 388700 + 65780$$

$C_T = 1299480$

(B) at least three face cards - Jack, King, or Queen 12 face cards

3 or 4 or 5

3 face and 2 non-face $_{12}C_3 \times {}_{40}C_2 = 220 \times 780 = 171600$
or
4 face and 1 non-face $_{12}C_4 \times {}_{40}C_1 = 495 \times 40 = 19800$
or
5 face and 0 non-face $_{12}C_5 \times {}_{40}C_0 = 792 \times 1 = 792$

$$171600 + 19800 + 792$$

192192

(C) at most 2 aces : 4 aces

0 or 1 or 2

0 aces and 5 non-aces $_4C_0 \times {}_{48}C_5 = 1712304$
or
1 ace and 4 non-aces $_4C_1 \times {}_{48}C_4 = 778320$
or
2 aces and 3 non-aces $_4C_2 \times {}_{48}C_3 = 103776$

$$1712304 + 778320 + 103776$$

C_T 2594400

(D) no spades 13 spades, 39 non-spades

0 spades and 5 nonspades

$$_{13}C_0 \times {}_{39}C_5 = 1 \times 575757 = 575757$$

2. A volleyball coach decides to use a starting line-up of ~~1 setter, 2 middle hitters,~~ 2 power hitters and 1 right side hitter. She chooses 14 players for the team: 3 setters, 4 middle hitters, 4 power hitters, 3 right side hitters. How many possible starting line-ups are there?

$${}_3C_1 \times {}_4C_2 \times {}_4C_2 \times {}_3C_1$$

$$= 3 \times 6 \times 6 \times 3 = 324$$

Textbook Questions: page 118 - 119; #1, 4, 5, 8, 10, 11, 13(A)