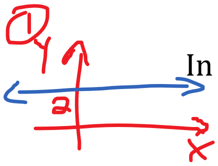
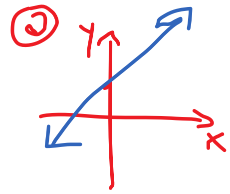
Math 3201

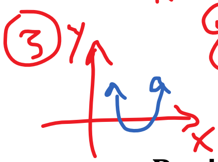
5.1 Introduction to the Graphs of Polynomials



In Math 1201/2201, we examined three types of polynomial functions:



- 1 **Constant Function** - horizontal line such as $y = 2$
- 2 **Linear Function** - sloped line, such as $y = 3x + 4$
- 3 **Quadratic Function** - parabola $y = x^2 + 5x - 2$



Review of Leading Coefficient and Degree of a Function

Degree: the highest exponent of x in a polynomial function.

Leading Coefficient: the number in front of the highest power of x in a polynomial function.

Example 1:
State the degree and leading coefficient for $y = -3x^2 + 4x - 1$

degree: 2

leading coefficient: −3

The following table summarizes the functions that we have previously studied:

Function	Degree	Type of Function	Graph
$f(x) = a$	0	constant	horizontal line
$f(x) = ax + b$	1	linear	line with slope a
$f(x) = ax^2 + bx + c$	2	quadratic	parabola

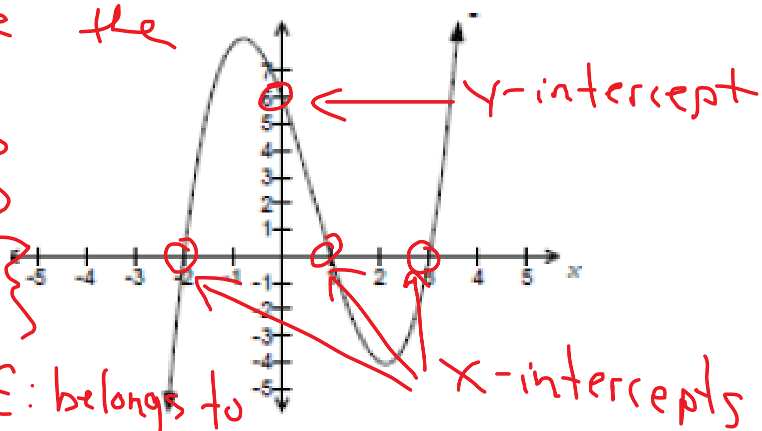
Introduction to Cubic Functions
A cubic function is one whose degree is 3. For example, $y = 4x^3 + 2x^2 - 5x + 9$. The graph of a cubic function often looks like a sideways "s".

All cubics have the following:

D: $\{x | x \in R\}$

R: $\{y | y \in R\}$

|: such that $\in$: belongs to

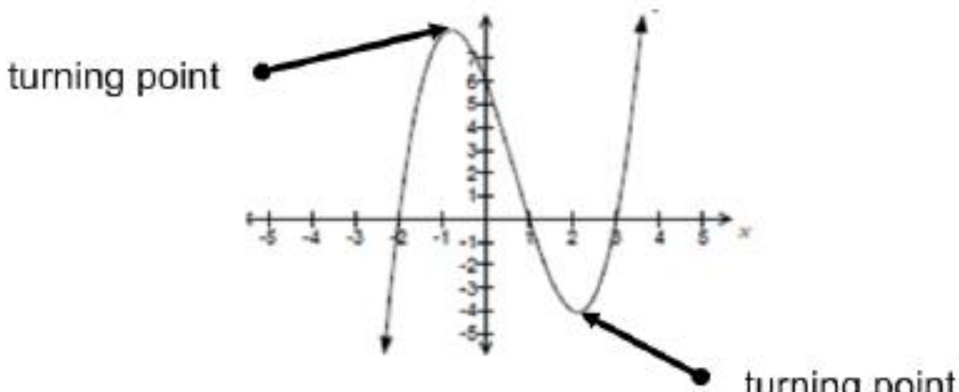


In order to examine the graphs of linear, quadratic, and cubic functions, there are several concepts we need to be familiar with.

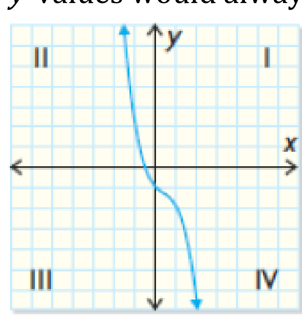
Intercepts: an x-intercept are the points where a graph intersects the x-axis, while a y-intercept is where a graph intersects the y-axis.

Domain and Range: domain refers to the set of x-values for a function, while the range refers to the set of y-values.

Turning Point: a point at which a graph changes direction. For example, a graph may be initially going upward, but after a turning point it will going downward.

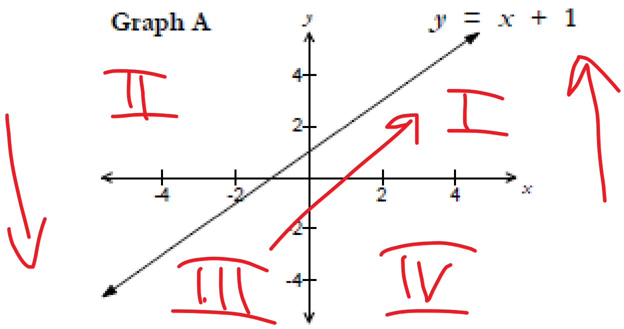


*Note – some graphs have no turning point. The y-values would always be increasing or decreasing. For example:



End Behavior: refers to what is happening at the two ends of a graph. More specifically at the left end where the x-values get smaller or at the right end where the x-values get larger.

Example 2:



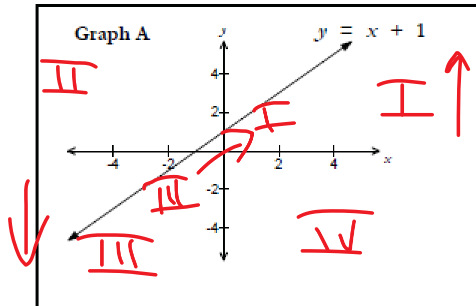
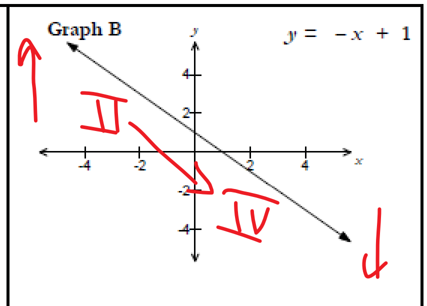
End Behaviour: falls to the left and rises to the right (i.e., $y \to -\infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$)

Line extends from Quadrant III to Quadrant I

Relating Leading Coefficient to End Behavior of a Function

Linear Functions: recall that the leading coefficient of a linear function is also the slope of the function. For example, for $y = 4x - 1$, the leading coefficient is 4 and the slope is also 4.

Use the following examples of linear functions to observe the relationship between leading coefficient and end behavior.

	
Slope: positive	Slope: negative
End Behaviour: falls to the left and rises to the right (i.e., $y \to -\infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$)	End Behaviour: falls to the right and rises to the left (i.e., $y \to -\infty$ as $x \to \infty$ and $y \to \infty$ as $x \to -\infty$)
Line extends from Quadrant III to Quadrant I	Line extends from Quadrant II to Quadrant IV

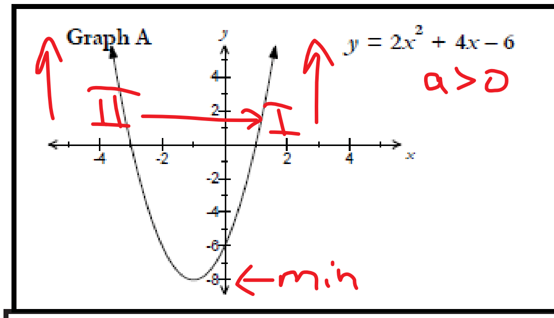
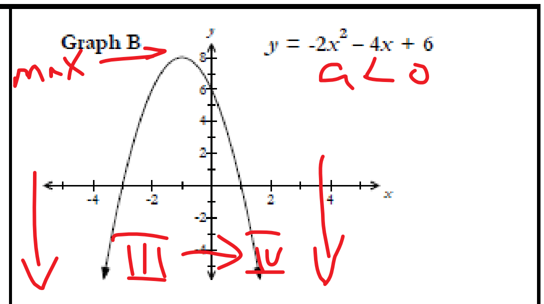
Summary

For a linear function:

- If the leading coefficient is positive, the graph will fall to the left and rise to the right.
- If the leading coefficient is negative, the graph will fall to the right and rise to the left.

Quadratic Functions

Use the following examples of quadratic functions to observe the relationship between leading coefficient and end behavior.

	
Leading coefficient is positive	Leading coefficient is negative
Range: $\{y \mid y \geq -8, y \in R\}$	Range: $\{y \mid y \leq 8, y \in R\}$
End Behaviour: rises to the left and rises to the right	End Behaviour: falls to the left and falls to the left
Parabola extends from Quadrant II to Quadrant I	Parabola extends from Quadrant III to Quadrant IV

Summary

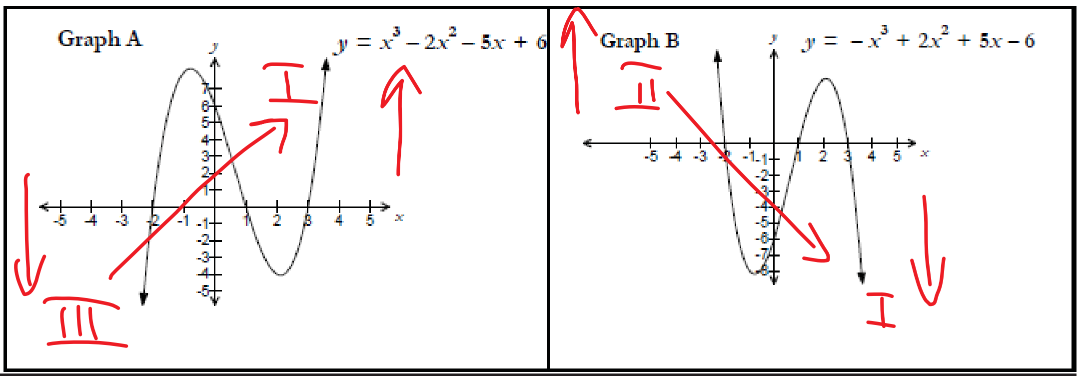
For a quadratic function:

- If the leading coefficient is positive, the graph will rise to the left and rise to the right, or rise on each side.
- If the leading coefficient is negative, the graph will fall to the right and fall to the left or falls on each side.

Notice that for quadratics, the end behavior is the same on both the left and right sides.

Cubic Functions

Use the following examples of cubic functions to observe the relationship between leading coefficient and end behavior.



Leading coefficient is positive	Leading coefficient is negative
Range: $\{y \mid y \in R\}$	Range: $\{y \mid y \in R\}$
End Behavior: falls to the left and rises to the right	End Behavior: rises to the left and falls to the right
Cubic extends from Quadrant III to Quadrant I	Cubic extends from Quadrant II to Quadrant IV

The standard equation for the cubic function is: $y = ax^3 + bx^2 + cx + d$

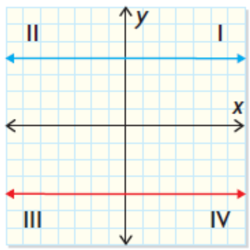
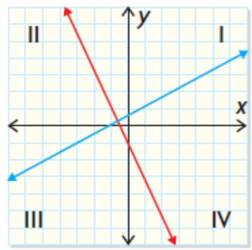
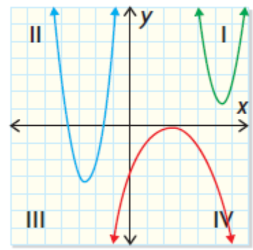
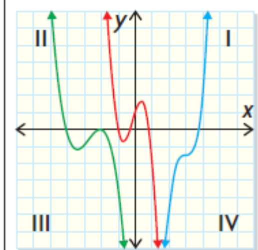
Summary

For a cubic function:

- If the leading coefficient is positive, the graph will fall to the left and rise to the right.
- If the leading coefficient is negative, the graph will fall to the right and rise to the left.

Notice that for cubic functions, the end behavior is opposite on the right and left sides of the graph. In other words, one end rises while the other falls.

The following summary table provides a more complete overview of linear, quadratic and cubic functions by addressing degree, intercepts, domain, range, end behavior and turning points.

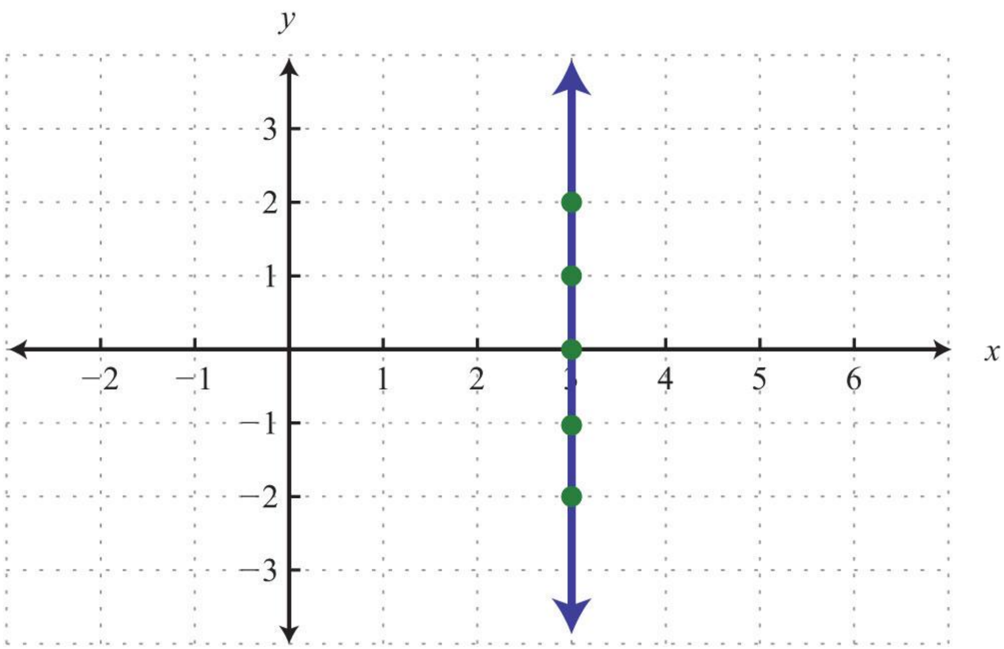
Type of Function	constant	linear	quadratic	cubic
Degree, *n*	0	1	2	3
Sketch				
Number of *x*-Intercepts	0, except for $y = 0$, for which every point is on the x-axis	1	0, 1, or 2	1, 2, or 3
Number of *y*-Intercepts	1	1	1	1
End Behaviour	Line extends from quadrant II to quadrant I or from quadrant III to quadrant IV.	Line extends from quadrant III to quadrant I or from quadrant II to quadrant IV.	Curve extends from quadrant II to quadrant I or from quadrant III to quadrant IV.	Curve extends from quadrant III to quadrant I or from quadrant II to quadrant IV.
Domain	$\{x \mid x \in R\}$	$\{x \mid x \in R\}$	$\{x \mid x \in R\}$	$\{x \mid x \in R\}$
Range	$\{y \mid y = \text{constant}, y \in R\}$	$\{y \mid y \in R\}$	$\{y \mid y \leq \text{maximum}, y \in R\}$ or $\{y \mid y \geq \text{minimum}, y \in R\}$	$\{y \mid y \in R\}$
Number of Turning Points	0	0	1	0 or 2

The following conclusions can be drawn based on the summary table:

- the graph of a polynomial function is continuous.
- the degree of the polynomial determines the shape of the function
- the maximum number of x-intercepts is equal to the degree of the function
- there is one y-intercept for all polynomial functions
- the graph of a polynomial function has only smooth turns
- a function of degree n has a maximum of $n - 1$ turns
- the end behavior of a line or curve is the becahvior of the y-values as x becomes large in the positive or negative direction

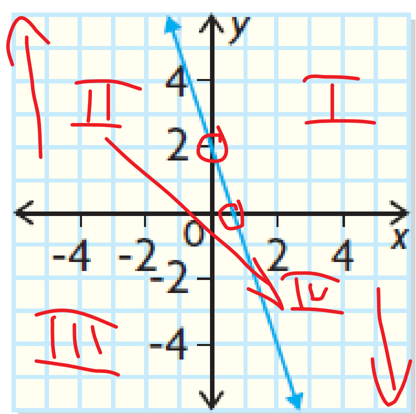
Vertical Lines

Vertical Lines are **not** polynomial functions since their graphs do **not** pass the vertical line test.



Example 3:

Determine whether the following graph is a function. Then state the x-intercepts, y-intercept, domain, range, end behavior, and the number of turning points.



X-int: $(0.75, 0)$

Y-int: $(0, 2)$

Domain: $\{x \mid x \in \mathbb{R}\}$

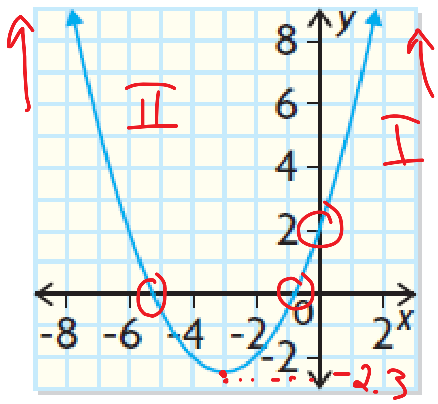
Range: $\{y \mid y \in \mathbb{R}\}$

End behavior: Rises left/falls right

Q II $\rightarrow$ Q IV

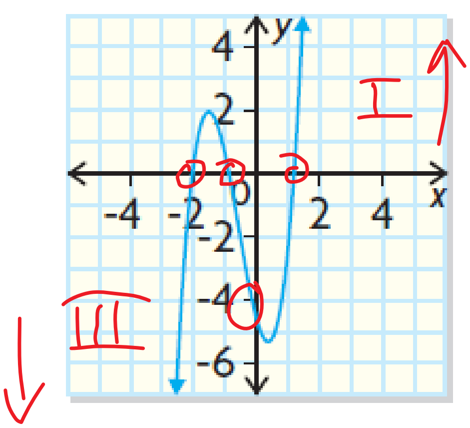
Turning points: 0

Example 4:
Determine whether the following graph is a function. Then state the x-intercepts, y-intercept, domain, range, end behavior, and the number of turning points.



X-int: $(-5.2, 0)$, $(-0.8, 0)$
y-int: $(0, 2)$
Domain: $\{x \mid x \in R\}$
Range: $\{y \mid y \geq -2.3, y \in R\}$
End Behavior: Rises left/Rises right
Q II $\rightarrow$ Q I
Turning points: 1

Example 5:
Determine whether the following graph is a function. Then state the x-intercepts, y-intercept, domain, range, end behavior, and the number of turning points.



X-int: $(-2, 0)$, $(-0.9, 0)$, $(1.1, 0)$
y-int: $(0, -4.3)$
Domain: $\{x \mid x \in R\}$
Range: $\{y \mid y \in R\}$
End behavior: Falls left/Rises right
Q III $\rightarrow$ Q I
Turning points: 2

Textbook Questions: page 277 #1 (omit f), 2 (omit f), 3 (just state whether each equation represent a linear, quadratic or cubic function), 4