Math 3201

6.3 Solving Exponential Equations

In Grade 9, Math 1201 and Math 2201, we looked at solving equations in which a variable was raised to an exponent.

Example 1:
Solve for x:

$$x^2 = 64$$

$\sqrt{x^2} = \sqrt{64}$ $x = \pm 8$

$(-8)^2 = 64$

$\sqrt{64} = -8$

Now, we will learn how to solve equations in which the variable is part of the exponent.

Recall from Math 1201 we learned about prime factorization and exponents. For example, since $2 \times 2 \times 2 = 8$, we can write 8 as 2^3. This logic is what you need to solve exponential equations. You might find it helpful to think about the definition of an exponent, and also to consider how this type of equation differs from the one we saw in the last example.

Consider the following comparison between the two examples:

	Definition of Exponent	Value of x
$x^2 = 64$	$x \times x = 64$	$x = \pm 8$
$2^x = 64$	$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$	$x = 6$

The first example $x^2 = 64$ is an example of a **polynomial function**. The variable x gets multiplied by itself two times to give 64.

The second example $2^x = 64$ is an example of an **exponential function**. The base 2 gets multiplied by itself "x" times to give 64.

Example 1:
Write as an exponent with lowest terms base:

(A) 16 $= 2^4$

4 4

2 2 2 2

(B) 27 $= 3^3$

$3 \cdot 9$

3 3 3

(C) 64 $= 2^6$

$8 \cdot 8$

$2\ 2\ 2\ 2\ 2\ 2$

(D) 125 $= 5^3$

$5 \cdot 25$

$5 \quad 5 \cdot 5$

A New Exponent Law

If we have two powers that are equal to each other and their bases are the same, then their exponents must be equal.

Case 1: The Bases are Already the Same

Example 2:

Solve for x:

(A)

$$3^x = 3^2$$

$$x = 2$$

(B)

$$5^6 = 5^x$$

$$x = 6$$

(C)

$$4^{x+1} = 4^6$$

$$x + 1 = 6$$

$$x = 6 - 1$$

$$x = 5$$

(D)

$$2^{2x+3} = 2^{4x-1}$$

$$2x + 3 = 4x - 1$$

$$3 + 1 = 4x - 2x$$

$$\frac{4}{2} = \frac{2x}{2}$$

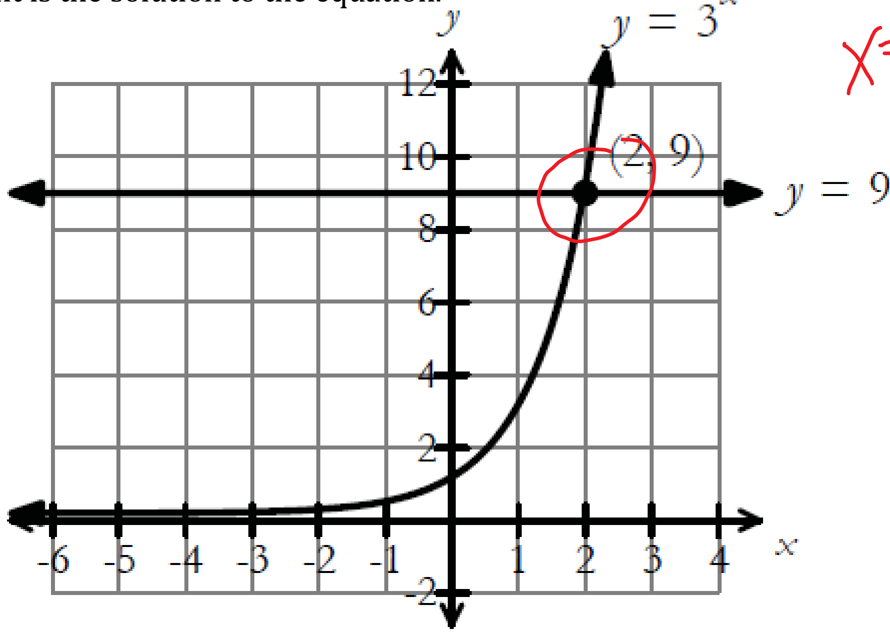
$$x = 2$$

Exponential equations could also be solved graphically:

Example 3:
Solve $3^x = 3^2$

$y=9$

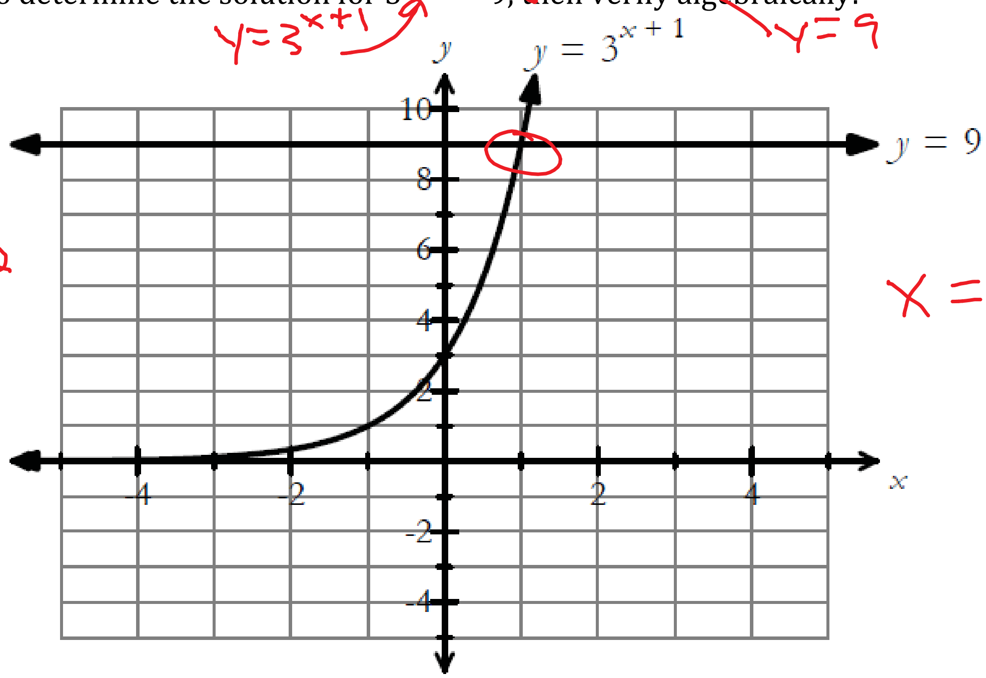
Graph $y = 3^x$ and $y = 3^2$ on the same grid and find the intersection point. The x-value of the intersection point is the solution to the equation.



$x=2$

Example 4:
Use the graph to determine the solution for $3^{x+1} = 9$, ~~then verify algebraically~~:

$y=3^{x+1}$ $y=9$



$3^{x+1} = 9$

$3^{x+1} = 3^2$

$x+1=2$

$x=2-1$

$x=1$

$x=1$

Case 2: The Bases are NOT Already the Same, but are Powers of One Another
Replace one or both base with a power, so that the powers on both sides of the equation will have the same base.

Example 5:
Solve for x:

(A)

$$2^{x-1} = 8$$

$8 = 2 \cdot 4 = 2 \cdot 2 \cdot 2 = 2^3$

$2^{x-1} = 2^3$

$x - 1 = 3$

$x = 3 + 1$

$x = 4$

(B)

$$3^{x-4} = 9^{x+1}$$

$3^{x-4} = 3^{2(x+1)}$

$3^{x-4} = 3^{2x+2}$

$x - 4 = 2x + 2$

$-4 - 2 = 2x - x$

$-6 = x$

or

$x = -6$

(C)

$$4^{2x+3} = 16^{4x-5}$$

$4^{2x+3} = 4^{2(4x-5)}$

$4^{2x+3} = 4^{8x-10}$

$2x + 3 = 8x - 10$

$3 + 10 = 8x - 2x$

$\frac{13}{6} = \frac{6x}{6}$

$x = 13/6$

(D)

$$4^{x+1} = 8^{3x-1}$$

$2^{2(x+1)} = 2^{3(3x-1)}$

$2^{2x+2} = 2^{9x-3}$

$2x + 2 = 9x - 3$

$2 + 3 = 9x - 2x$

$5 = 7x$

$x = \frac{5}{7}$

Exponential Equations Involving Radicals
Replace the radical with an exponent.

Recall that

$$\sqrt{x} = x^{\frac{1}{2}},\ \sqrt[3]{x} = x^{\frac{1}{3}},\ \sqrt[4]{x} = x^{\frac{1}{4}} \ldots$$

$\sqrt[n]{x} = x^{\frac{1}{n}}$

$\sqrt[2]{x} = \sqrt{x}$

Example 6:
Solve for x:

$$\sqrt{8} = 2^{3x-4}$$

$8^{\frac{1}{2}} = 2^{3x-4}$

$2^{3\left(\frac{1}{2}\right)} = 2^{3x-4}$

$2^{\frac{3}{2}} = 2^{3x-4}$

$\cancel{2} \cdot \frac{3}{\cancel{2}} = (3x-4)2$

$3 = 6x - 8$

$3 + 8 = 6x$

$\frac{11}{6} = \frac{6x}{6}$

$x = 11/6$

Exponential Equations Involving Fractions
Replace the fraction with a power involving a negative exponent .

Recall that

$$\frac{1}{x} = x^{-1}$$

$\frac{1}{x^n} = x^{-n}$ $\quad x^0 = 1$

$\frac{1}{x^{-n}} = x^n$

Example 7:
Solve for x:

$$\frac{1}{9} = 3^{x+4}$$

$\frac{1}{3^2} = 3^{x+4}$

$3^{-2} = 3^{x+4}$

$-2 = x + 4$

$-2 - 4 = x$

$x = -6$

or

$9^{-1} = 3^{x+4}$

$3^{2(-1)} = 3^{x+4}$

$3^{-2} = 3^{x+4}$

$\vdots$

Power is Multiplied by a Constant
Divide each side by the constant and then solve the resulting exponential equation.

Example 8:
Solve for x:

$$\frac{\cancel{4}(2)^{x-1}}{\cancel{4}} = \frac{8}{4}$$

$2^{x-1} = 2^1$

$x - 1 = 1$

$x = 1 + 1$

$x = 2$

common mistake

$$\cancel{4(2)^{x-1} = 8}$$

$$\cancel{8^{x-1} = 8}$$

Wrong!!!

Try these:

(A)

$$5^{2x+1} = 25$$

$5^{2x+1} = 5^2$

$2x + 1 = 2$

$2x = 2 - 1$

$\frac{2x}{2} = \frac{1}{2}$

$x = 1/2$

(B)

$$2^x = 128$$

$2^x = 2^7$

$x = 7$

or

128

$2 \cdot 64$

$2 \quad 8 \quad 8$

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

2^7

(C)

$$27^{4x} = 9^{x+1}$$

$3^{3(4x)} = 3^{2(x+1)}$

$\cancel{3}^{12x} = \cancel{3}^{2x+2}$

$12x = 2x + 2$

$12x - 2x = 2$

$10x = 2$ → $x = \frac{2}{10}$

$x = \frac{1}{5}$

(D)

$$\sqrt{5} = 25^{3x+4}$$

$5^{\frac{1}{2}} = 5^{2(3x+4)}$

$2 \cdot \frac{1}{2} = (6x + 8) \cdot 2$

$1 = 12x + 16$

$1 - 16 = 12x$

$-15 = 12x$ → $x = \frac{-15}{12}$

$x = -\frac{5}{4}$

(E)

$$5^{2x-2} = \left(\frac{1}{25}\right)^{x-1}$$

$5^{2x-2} = \left(\frac{1}{5^2}\right)^{x-1}$

$5^{2x-2} = 5^{-2(x-1)}$

$5^{2x-2} = 5^{-2x+2}$

$2x - 2 = -2x + 2$

$2x + 2x = 2 + 2$ → $\frac{4x}{4} = \frac{4}{4}$

$x = 1$

(F)

$$\frac{1}{8} = 2^{x-5}$$

$8^{-1} = 2^{x-5}$

$2^{3(-1)} = 2^{x-5}$

$2^{-3} = 2^{x-5}$

$-3 = x-5 \rightarrow -3+5 = x$

$x = 2$

(G)

$$16^{2x+1} = \left(\frac{1}{2}\right)^{x-3}$$

$2^{4(2x+1)} = 2^{-1(x-3)}$

$2^{8x+4} = 2^{-x+3}$

$8x+4 = -x+3$

$8x+x = 3-4$

$9x = -1 \rightarrow x = -\frac{1}{9}$

(H)

$$\frac{3(9)^{5x}}{3} = \frac{27}{3}$$

$9^{5x} = 9^1$

$\frac{5x}{5} = \frac{1}{5} \rightarrow x = \frac{1}{5}$

Textbook Questions: page 361 - 363 #2, 4, 5, 7 (solve all algebraically)