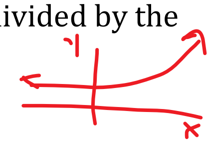
Math 3201

6.3C Applications of Exponential Functions

How do we know if a set of data can be modeled using an exponential function? Whenever a quantity changes by the same factor, in other words, gets multiplied or divided by the same value, each time, then it can be modeled by an exponential function.

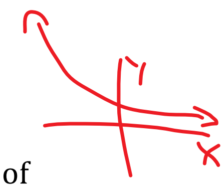
exponential growth



Some examples are:

- a population doubles each year.
- the amount of money in a bank account increases by 0.1% each month
- the mass of a radioactive substance decreases by ½ every 462 years.

exponetial decay



Half-Life Problems

In a half-life problem, the amount of a substance decreases by ½ every fixed number of years. The formula is:

$y=a(b)^{\frac{x}{c}}$

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

"A not"

Where:

- A_0 represents the initial amount of the substance that was present.
- t represents the time
- h represents the half-life or the amount of time taken for the substance to decrease by $\frac{1}{2}$.
- $A(t)$ is the amount of the substance present at time t.

Note: Some other applications might have an equation given with a number other than $\frac{1}{2}$ as the base in the power. For example, if a quantity doubles, the base will be 2, if it triples, the base will be 3, and so on.

Example 1:

When diving under water, the light decreases as the depth of the diver increases. On a sunny day off the coast of Vancouver Island, a diving team recorded 100% visability at the surface but only 25% visability 10 m below the surface. The team determined that the visability for the dive could be modelled by the following function:

$$A(x) = A_0\left(\frac{1}{2}\right)^{\frac{x}{h}}$$

A_0 represents the percentage of light at the surface of the water, x represents the depth in meters, h represents the depth in meters at which there is only half the original visibility, and $A(x)$ represents the percentage of light at a depth of x meters. At what depth will the visibility be only half of what it was at the surface?

$A_0 = 100$

$A(x) = 25$

$x = 10$

$h = ?$

$$\frac{25}{100} = \frac{100\left(\frac{1}{2}\right)^{\frac{10}{h}}}{100}$$

$$\frac{1}{4} = \left(\frac{1}{2}\right)^{\frac{10}{h}}$$

$$\frac{1}{2^2} = \left(\frac{1}{2}\right)^{\frac{10}{h}}$$

$$\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^{\frac{10}{h}}$$

$$\frac{2}{1} = \frac{10}{h}$$

$$\frac{2h}{2} = \frac{10}{2}$$

$$h = 5$$

At 5m, visability will be half of visability at the surface.

Example 2:

The radioactive element cobalt-60 can be used to treat cancer patients. Radioactive elements decay into other elements in a predictable way over time. The percent of cobalt-60, $A(t)$, left in a sample can be modelled by the half-life function:

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{5.3}}$$

Where t represents the time, in years, after the initial time and A_0 represents the initial amount, 100%, of the cobalt-60.

(A) How long does it take for the sample of cobalt-60 to reduce to half its initial amount?

$A_0 = 100$

$A(t) = 50$

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{5.3}}$$

$$\frac{50}{100} = \frac{100\left(\frac{1}{2}\right)^{\frac{t}{5.3}}}{100}$$

$$\frac{1}{2} = \frac{1}{2}^{\frac{t}{5.3}}$$

$$5.3 \cdot 1 = \frac{t}{\cancel{5.3}} \cdot \cancel{5.3}$$

$$t = 5.3$$

(B) What percent of cobalt-60 will remain in a sample after 10 years? Round to the nearest percent.

$A_0 = 100$

$t = 10$

$A(t) = ?$

$$A(t) = 100\boxed{\left(\frac{1}{2}\right)^{\frac{10}{5.3}}}$$

$\leftarrow$ calculate first

$(1 \div 2)\boxed{\wedge}(10 \div 5.3)$

$$A(t) = 100(0.2704)$$

$$A(t) = 27.04\%$$

There will be 27.04% of cobalt-60 after 10 years.

(C) Determine how long it will take, to the nearest year, until only 25% of cobalt-60 remains?

$A(t) = 25$

$A_0 = 100$

$$\frac{25}{100} = \frac{100\left(\frac{1}{2}\right)^{\frac{t}{5.3}}}{100}$$

$$\frac{1}{4} = \frac{1}{2}^{\frac{t}{5.3}}$$

$$\frac{1}{2^2} = \frac{1}{2}^{\frac{t}{5.3}}$$

$$\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^{\frac{t}{5.3}}$$

$$5.3 \cdot 2 = \frac{t}{\cancel{5.3}} \cdot \cancel{5.3}$$

$$t = 10.6$$

25% of cobalt-60 remains after 10.6 years.

Other Applications

Example 3:

The population of trout growing in a lake can be modelled by the function $P(t) = 200(2)^{\frac{t}{5}}$ where $P(t)$ represents the number of trout and t represents the time in years after initial count. How long will it take for there to be 6400 trout?

$t = ?$

$P(t) = 6400$

$$\frac{6400}{200} = \frac{200(2)^{\frac{t}{5}}}{200}$$

$$32 = 2^{\frac{t}{5}}$$

$$2^5 = 2^{\frac{t}{5}}$$

$$5 \cdot 5 = \frac{t}{\cancel{5}} \cdot \cancel{5}$$

$$t = 25$$

It will take 25 years to get 6400 trout.

Example 4:

Small rural water systems are often contaminated with bacteria by animals. Suppose that a water tank is infested with a colony of 14,000 E. coli bacteria. In this tank, the colony doubles in number every 4 days. The number of bacteria present in the tank after t days can be modeled by the function:

$$A(t) = 14000(2)^{\frac{t}{4}}$$

Determine the value of t when $A(t) = 224\,000$. What does your answer mean in this context? Explain.

$$\frac{224000}{14000} = \frac{14000(2)^{\frac{t}{4}}}{14000}$$

$$16 = 2^{\frac{t}{4}}$$

$$2^4 = 2^{\frac{t}{4}}$$

$$4 \cdot 4 = \frac{t}{\cancel{4}} \cdot \cancel{4}$$

$\Rightarrow t = 16$

It takes 16 days to get 224 000 bacteria.

Example 5:
The half-life of a radioactive isotope is 30 hours. The amount of radioactive isotope $A(t)$, at time t, can be modelled by the function:

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

Determine algebraically how long it will take for a sample of 1792 mg to decay to 56 mg.

$h = 30$

$A_0 = 1792$

$A(t) = 56$

$t = ?$

$$\frac{56}{1792} = \frac{1792\left(\frac{1}{2}\right)^{\frac{t}{30}}}{1792}$$

$$\frac{1}{32} = \frac{1}{2}^{\frac{t}{30}}$$

$$\frac{1}{2^5} = \frac{1}{2}^{\frac{t}{30}}$$

$$\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^{\frac{t}{30}}$$

$$30 \cdot 5 = \frac{t}{\cancel{30}} \cdot \cancel{30}$$

$$t = 150$$

It will take 150 hours to decay.

Textbook Questions: page 363 – 365 #11, 15, 16