Math 3201

6.5 Financial Applications Involving Exponential Functions

When you invest money, your money earns **interest**. That is, the amount of money that you will have after a period of time will actually be greater than the amount that you invested. The extra money, above the amount you invested, is called **interest**.

The original amount of money that you invest is called the **principal**.

There are two types of interest that we will learn about:

- **Simple Interest:** the amount of interest that you earn is calculated **only** based on the amount of money that you invested. That is, you only earn interest on the money that you invested.

- **Compound Interest:** interest is earned on two different things:
  (i) the amount of money that you invest
  (ii) the interest that you earn on the money that you invested

Part 1: Simple Interest

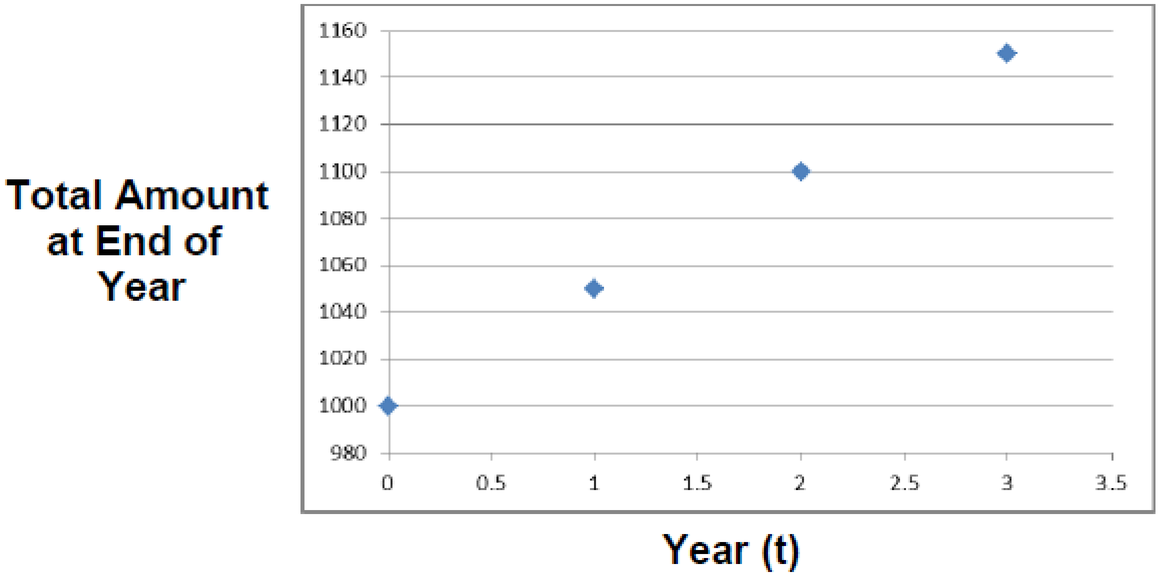
Suppose someone invests \$1000 into a savings account that earns an annual interest of 5%.

Suppose this account has a simple interest rate. That means that interest is applied only on the original amount each year. The following table shows the amount of money in the account after the first three years.

Year (t)	Total Amount at the End of the Year (A)
0	\$1000
1	\$1050 (\$1000 + \$1000(0.05)(1))
2	\$1100 (\$1000 + \$1000(0.05)(2))
3	\$1150 (\$1000 + \$1000(0.05)(3))

Notice that each year, an extra 5% of the original amount gets added on.

Consider what this data would look like on a graph:



Notice that the data appears to be linear. This is because we are adding on the same amount of money each year which is 5% of the original amount.

Notice from the table of values that the amount present in the saving account at the end of each year is made up of the principal amount or the original amount invested, plus 5% of the principal per year, for each year that has passed by. We can write this as an equation:

$$A = P(1 + rt) \text{ or } A = P + Prt$$ ← provided on assessments.

Where: A represents the amount present
P represents the principal amount
r = interest percentage divided by 100
t represents the number of years

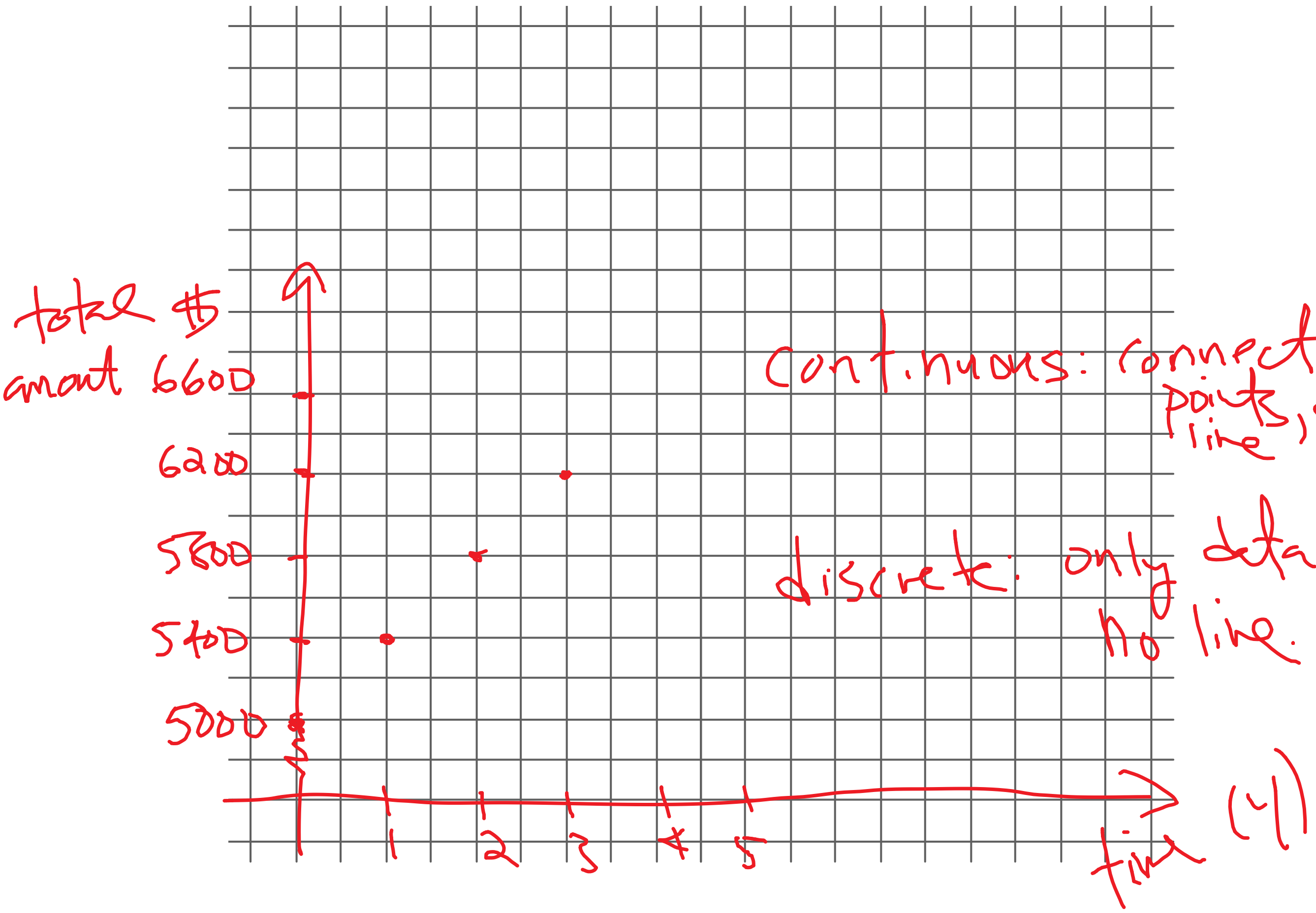
Example 1:
Kyle invested his summer earnings of $5000 at 8% simple interest, paid annually. (0.08) (Gives yearly total.)

(A) Complete a table of values and graph the growth of the investment for 3 years using "time (years)" as the domain and "value of the investment" as the range.

Year (t)	Total Amount at End of Year (A)
0	$5000
1	5000 + 400 = $5400
2	5400 + 400 = $5800
3	5800 + 400 = $6200

5000 × 0.08 = $400

$A = P(1 + rt)$
$A = 5000(1 + 0.08(1))$
$A = 5000(1.08)$
$A = 5400$



continuous: connect data points, draw line

discrete: only data points, no line.

(B) What does the shape of the graph tell you about the type of growth? Why is the data discrete?

Linear. Increases by the same amount each year.

Discrete because only make increments of $400.

(C) What do the y-intercept and slope represent for the investment?

y-intercept: principle amount

slope: interest

(D) What is the value of the investment after 10 years?

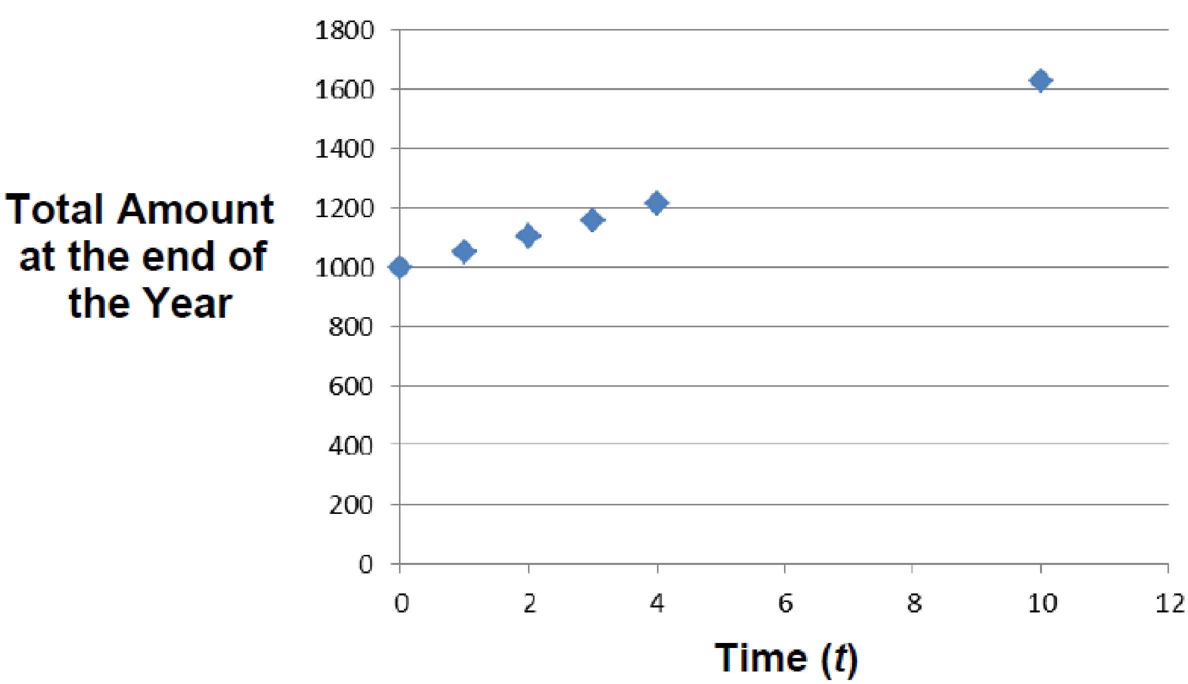
$t = 10$

$A = P(1 + rt) = 5000[1 + 0.08(10)] = \$9000$

Part 2: Compound Interest
Suppose someone invests \$1000 at 5% compound interest. This means that the principal \$1000 earns interest each year, and that the interest earned also earns more interest.

Year (t)	Amount of Annual Interest	Total Amount at the End of the Year (A)
0		\$1000
1	$1000 \times 0.05 = 50$	\$1050 $(1000(1.05)^1)$
2	$1050 \times 0.05 = 52.50$	\$1102.50 $(1000(1.05)^2)$
3	$1102.50 \times 0.05 = 55.13$	\$1157.63 $(1000(1.05)^3)$
4	$1157.63 \times 0.05 = 57.88$	\$1215.51 $(1000(1.05)^4)$

Notice that each year, the amount increases by an even greater amount than it increased by in the previous year. We can see this from the graph:



An extra point was added in on this graph to show that the pattern here is not linear, but is **exponential**.

Recall the format for an exponential function: $y = a(b)^x$. Think about what each value in the equation represents:

- y is the dependent variable, which in this case is A is the amount present.
- a is the initial amount, which in this case is $1000.
- b is the value by which the dependent variable gets multiplied by each year, which in this case is 1.05. Confirmed by looking at the table of values.
- x is the independent variable, which in this case is the year.

Putting all this together, we get the following formula for compound interest for the above problem.

$$A = 1000(1.05)^t$$

within the context of this problem, the b value equals 1 plus the interest rate, as a decimal. That is, 1 + i.

In general, the compound interest can be found using the formula:

$$A = P(1 + i)^n$$

given on assessments

Where:
- P is the the principle amount
- i is the interest rate per compounding period
- n is the number of compounding periods

Notice that i is the interest rate **per compounding period**. In the example we looked at, the interest rate was 5% compounded annually, or every 1 year. Thus,

$$i = \frac{0.05}{1} = 0.05$$

In general,

$$i = \frac{\text{annual rate}}{\text{number of times interest is paid}}$$

Compounding periods are usually daily, weekly, semimonthly, monthly, quarterly, semiannually or annually. The table below shows how many times interest is paid, and the interest rate for each of these options.

Compounding Period	Number of Times Interest Is Paid	Interest Rate per Compounding Period, i
daily	365 times per year	$i = \frac{\text{annual rate}}{365}$
weekly	52 times per year	$i = \frac{\text{annual rate}}{52}$
semi-monthly	24 times per year	$i = \frac{\text{annual rate}}{24}$
monthly	12 times per year	$i = \frac{\text{annual rate}}{12}$
quarterly	4 times per year	$i = \frac{\text{annual rate}}{4}$
semi-annually	2 times per year	$i = \frac{\text{annual rate}}{2}$
annually	1 time per year	$i = \frac{\text{annual rate}}{1}$

Example 2:
Suppose an investment of $1000 has a compound interest rate of 6% compounded semiannually. Write an exponential equation representing the situation and use the equation to determine how much money will be in the account after 6 years.

$$i = \frac{0.06}{2} = 0.03$$

$$n = \frac{2 \text{ times}}{1 \text{ year}} \times 6 \text{ years} = 12$$

$$P = 1000(1+0.03)^{12} = 1000(1.03)^{12} = \$1425.76$$

Potential Constructed Response question on public exam.

Example 3:
A principal of $3000 was invested for 5 years in an account in which 8% compound interest was compounded quarterly. How much money was in the account at the end of the 5 years?

$P = 3000$

$i = \frac{0.08}{4} = 0.02$

$n = 4 \times 5 = 20$

$A = P(1+r)^n$

$A = 3000(1.02)^{20}$

$A = \$4457.84$

Example 4:
Consider the following statements:

- Emily invested $3000 for a term of 5 years with a simple interest rate of 4% per year, paid annually.
- Zachary invested $3000 for a term of 5 years with a compound interest rate of 4% per year, paid annually.

Which investment results in the greatest value at 5 years?

Emily

$P = 3000$

$t = 5$

$i = 0.04$

$A = 3000[1 + 0.04(5)]$

$A = \$3600$

Zack

$P = 3000$

$n = 5(1) = 5$

$i = \frac{0.04}{1}$

$P = 3000(1.04)^5$

$P = \$3649.96$

Zack's investment is greater.

Example 5:
$3000 was invested at 6% per year compounded monthly. Write the exponential function in the form $A = P(1+i)^n$. What will be the future value of the investment after 4 years?

$P = 3000$

$i = \frac{0.06}{12} = 0.005$

$n = 4 \cdot 12 = 48$

$A = 3000(1 + 0.005)^{48}$

$A = 3000(1.005)^{48}$

$A = \$3811.47$

Example 6:
$2000 is invested at 6% per year compounded semiannually. Carol defined the exponential function as $A = 2000(1.03)^n$ where n is the number of 6 month periods. Is Carol's reasoning correct? Why or why not?

$P = 2000$

$i = \frac{0.06}{2} = 0.03$

$A = 2000(1.03)^n$

Yes, Carol is correct.

Example 7:
Joe invests \$5000 into a high interest savings bond that has an annual interest rate of 9% compounded monthly. Ask students to answer the following:

(A) Write the exponential equation $A = P(1+i)^n$ that represents the situation.

$P = 5000$

$i = \frac{0.09}{12} = 0.0075$

$A = 5000(1.0075)^n$

(B) How much will the bond be worth after 1 year?

$n = 12 \cdot 1 = 12$

$A = 5000(1.0075)^{12}$

$A = \$5469.03$

Example 8:
\$1000 is invested at 8.2% per year for 5 years. Using the equation $A = P(1+i)^n$, ask students to determine the account balance if it is compounded annually, quarterly, monthly and daily. Would it make more of a difference if the interest is accrued for 25 years? Explain your reasoning.

Annually:

$i = \frac{0.082}{1} = 0.082$

$n = 1 \times 5 = 5$

$A = 1000(1.082)^5$

$A = \$1482.98$

Quarterly:

$i = \frac{0.082}{4} = 0.0205$

$n = 4 \times 5 = 20$

$A = 1000(1.0205)^{20}$

$A = \$1500.58$

Monthly:

$i = \frac{0.082}{12} = 0.0068$

$n = 12 \times 5 = 60$

$A = 1000(1.0068)^{60}$

$A = \$1501.73$

Daily:

$i = \frac{0.082}{365} = 0.000225$

$n = 365 \times 5 = 1825$

$A = 1000(1.000225)^{1825} = \$1507.69$

25 years

annual: $n = 1 \times 25 = 25$

$A = 1000(1.082)^{25}$

$A = \$7172.68$

daily: $n = 365 \times 25 = 9125$

$A = 1000(1.000225)^{9125}$

$A = \$7790.41$

Example 9:
\$2000 is invested for three years that has an annual interest rate of 9% ***compounded monthly***. Lucas solved the following equation $A = 2000(1.0075)^3$. Identify the error that Lucas made. Correct the error and solve the problem.

$P = 2000$

$i = \frac{0.09}{12} = 0.0075$

$\boxed{n = 3 \times 12 = 36}$ ← error

$A = 2000(1.0075)^{36}$

$A = \$2617.29$