Math 3201

8.4 The Equations of Sinusoidal Functions

In Math 2201, we examined the quadratic function $y = a(x - h)^2 + k$, and learn how to read properties such as vertex, direction of opening, etc. from the equation.

Here we will examine transformations of the sine and cosine function, and learn how to read various properties from the equation.

Transformed sinusoidal equations are written in the following forms:

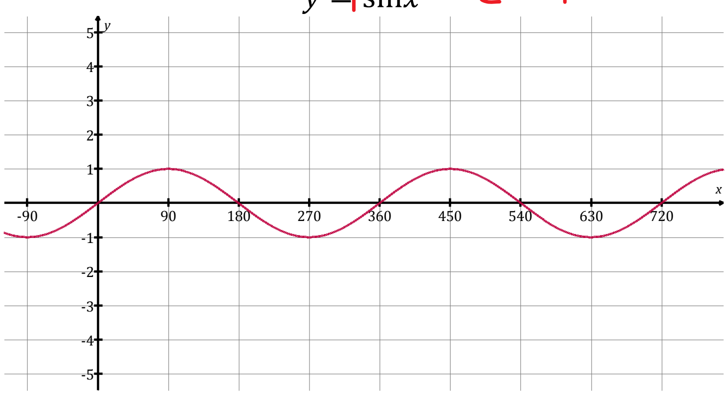
$$y = a \sin b(x - c) + d$$

$$y = a \cos b(x - c) + d$$

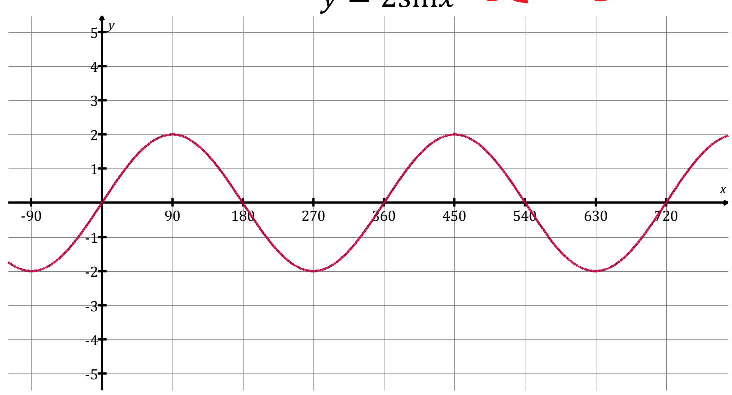
Examining the Impact of the "a" Value on the Graph of a Sinusoidal Function

Example 1:
Consider the graphs shown for each equation:

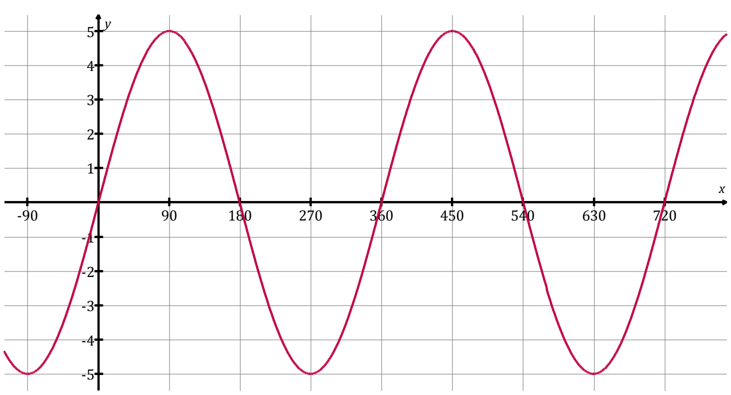
$y = 1\sin x$ $a = 1$



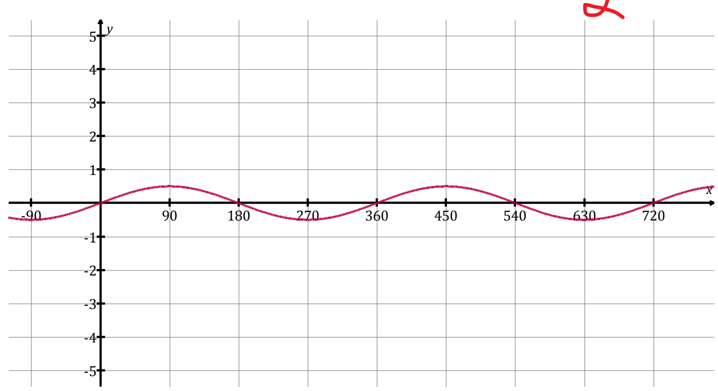
$y = 2\sin x$ $a = 2$



$y = 5\sin x$ $a = 5$



$y = \frac{1}{2}\sin x$ $a = \frac{1}{2}$



(A) What happens to the amplitude if a gets larger?

The amplitude gets larger.

(B) Is the shape of the graph affected by the parameter a?

yes. The amplitude changes. The graph stretches, vertically.

(C) How is the range affected by the parameter a?

The a-value affects the max and min value because the amplitude changes.

(D) Will the value of a affect the cosine graph in the same way that it affects the sine graph? Why or why not?

Yes, because both graphs are sinusoidal.

Summary of the "a" Value

The a value stretches or compresses a graph vertically. It equals the amplitude of the function.

$$a = \text{amplitude}$$

Example 2:
Determine the a value and state the amplitude for each equation:

(A) $y = \overset{a}{2}\sin\overset{b}{3}(x - \overset{c}{90^o}) + \overset{d}{1}$

$a=2$ $\therefore$ amplitude is 2.

(B) $y = \overset{a}{0.75}\sin\overset{b}{2}(x + \overset{c}{45^o}) - \overset{d}{3}$

$a=0.75$ or $\frac{3}{4}$ $\therefore$ amplitude is 0.75.

Examining the Impact of the "*d*" Value on the Graph of a Sinusoidal Function

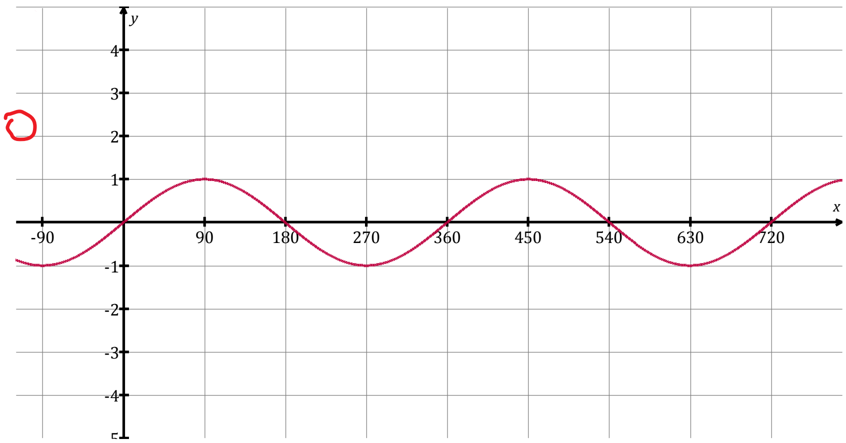
Example 3:
Consider the graphs shown for each equation:

i. $y = \sin x$

Think: $y = \sin x + 0$

$d = 0$

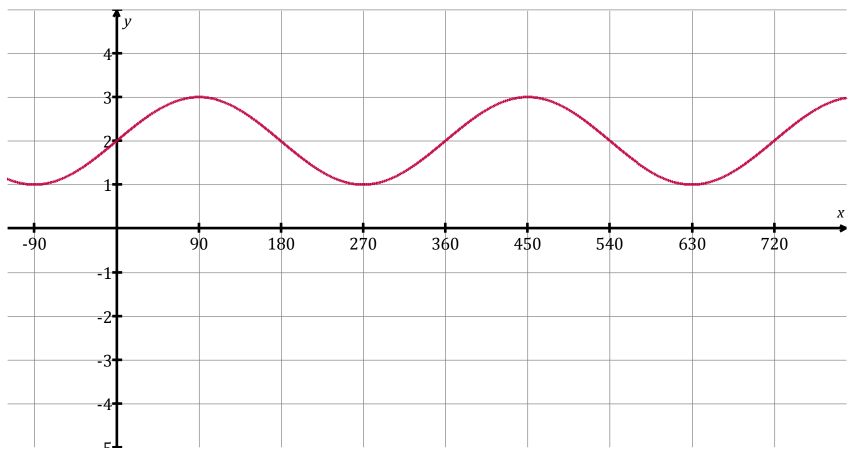
midline: $y = 0$



ii. $y = \sin x + 2$

$d = 2$

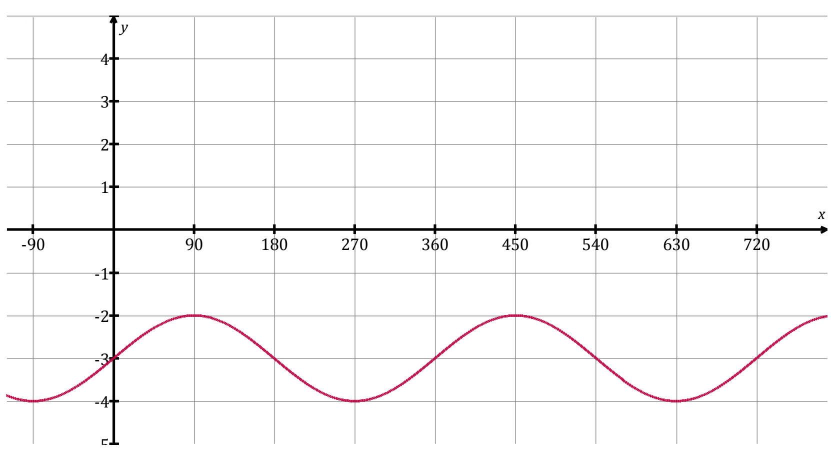
midline: $y = 2$



iii. $y = \sin x - 3$

$d = -3$

midline: $y = -3$



(A) How does each graph change when compared to $y = \sin x$?

Translates (shifts) vertically (up or down).

(B) How is the value of $\boldsymbol{d}$ related to the equation of the **midline**?

The d-value is the equation of the midline.
$y = d$

(C) Is the shape of the graph or the location of the graph affected by the parameter d?

Shape is not affected. Location is affected.

(D) Is the period affected by changing the value of d?

No.

(E) Will the value of d affect the cosine graph in the same way that it affects the sine graph? Why or why not?

Yes. See example 1(E).

Summary of the "*d*" Value
The "d" value gives us the equation of the midline of a sinusoidal function.

$$d = \text{Equation of Midline}$$

Example 4:
Identify the equation of the midline for each sinusoidal function:

(A) $y = \overset{a}{2}\sin\overset{b}{3}(x - \overset{c}{90^o}) + \overset{d}{1}$

$d = 1$
$y = 1$

(B) $y = 0.75\sin 2(x + 45^o) - 3$

$d = -3$
$y = -3$

Examining the Impact of the "b" Value on the Graph of a Sinusoidal Function

Example 5:
Consider the graphs shown for each equation:

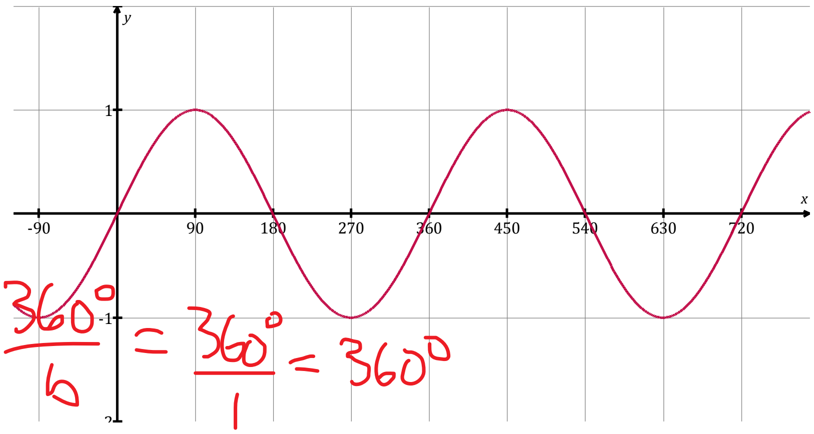
i. $y = \sin x$

b

$b = 1$

period $= 360°$

equation: period $= \frac{360°}{b} = \frac{360°}{1} = 360°$

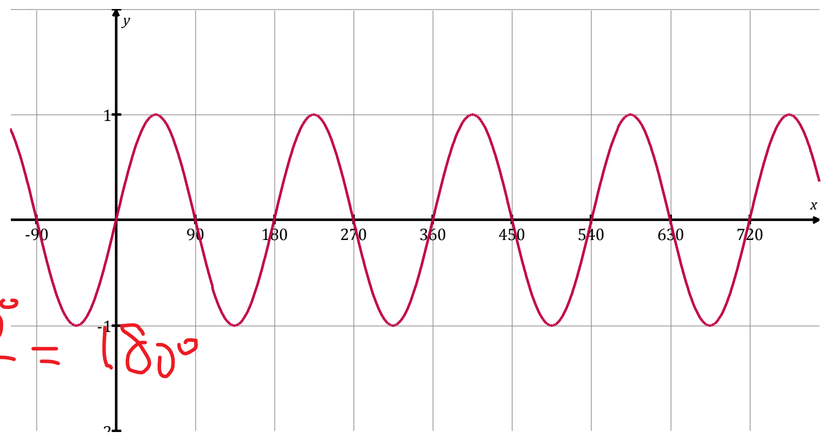


ii. $y = \sin 2x$

$b = 2$

period $= 180°$

equation: period $= \frac{360°}{2} = 180°$

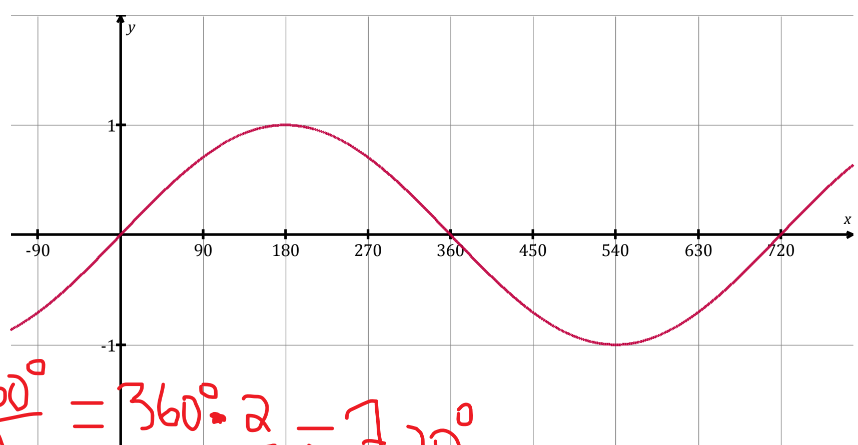


iii. $y = \sin \frac{1}{2}x$

$b = \frac{1}{2}$

period $= 720°$

equation: period $= \frac{360°}{\frac{1}{2}} = 360° \cdot \frac{2}{1} = 720°$



(A) How does each graph change when compared to $y = \sin x$?

period changes.

(B) How is the value of b related to the period?

see summary below

(C) Write an equation that relates period to the b value.

Summary of the "*b*" Value

The "b" value gives us the period of a sinusoidal function.

In degrees:

$$period = \frac{360^o}{b}$$

$360^\circ = 2\pi$

In radians:

$$period = \frac{2\pi}{b}$$

Example 6:
Determine the period for each equation:

(A) $y = \overset{a}{2}\sin\overset{b}{3}(x - \overset{c}{90^o}) + \overset{d}{1}$ $b=3$

$period = \frac{360^\circ}{3} = 120^\circ$

(B) $y = 0.5\cos 2(x + 180^o) + 2$ $b=2$

$period = \frac{360^\circ}{2} = 180^\circ$

(C) $y = 4\cos 3\left(x - \frac{\pi}{4}\right) - 5$ $b=3$

$period = \frac{2\pi}{3}$

Examining the Impact of the "*c*" Value on the Graph of a Sinusoidal Function

Example 7:
Consider the graphs shown for each equation:

i. $y = \sin x$

Think: $y = \sin(x - 0^\circ)$

$c = 0^\circ$

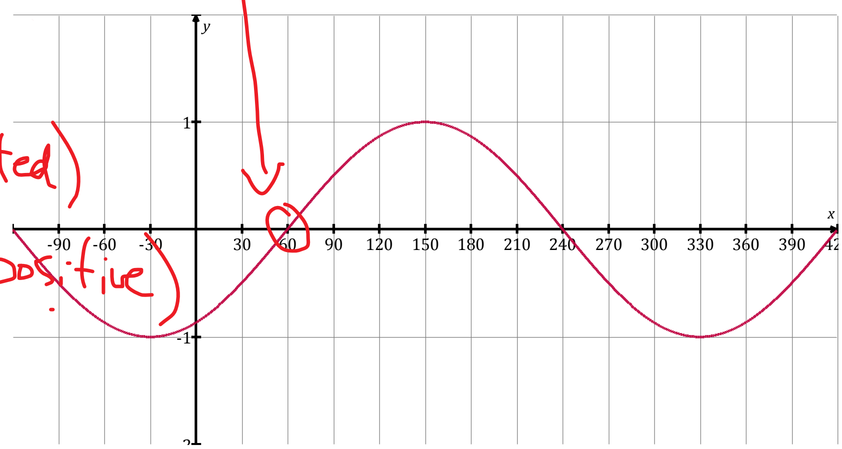
Starting point: 0°

ii. $y = \sin(x - 60^\circ)$

$c = 60^\circ$

graph translated (shifted) 60° to the right. (positive)

Starting point: 60°

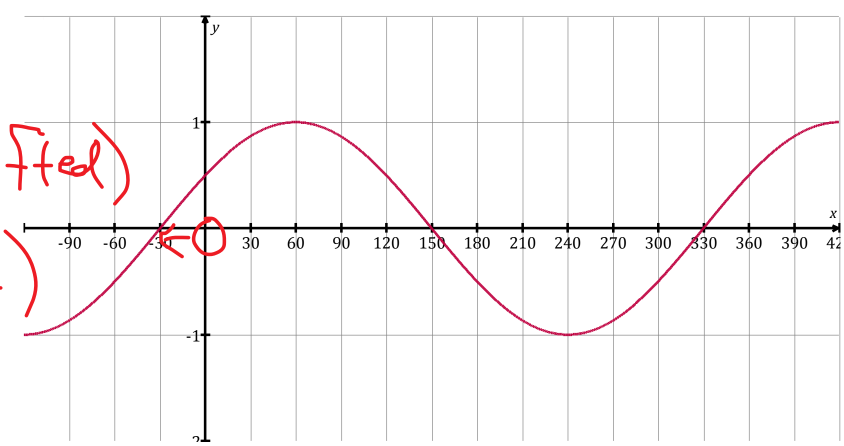


iii. $y = \sin(x + 30^\circ)$

$c = -30^\circ$

graph translated (shifted) 30° left (negative)

Starting point: -30°



(A) How does each graph change when compared to $y = \sin x$?

Translates (shifts) horizontally.

(B) How is the value of c related to starting point?

C is the starting point, once you change the sign from the equation.

(C) How can we determine the "phase shift" from the equation of the sinusoidal equation?

Phase shift = horizontal translation. Take the opposite sign from the equation.

Summary of the "c" Value

The "c" value **shifts a graph horizontally**. The shift is obtained by taking the opposite sign of the number after x in the equation.

Examples 8:
Determine the horizontal shift for each equation:

(A) $y = \underset{a}{2}\sin\underset{b}{3}(x - \underset{c}{90^o}) + \underset{d}{1}$

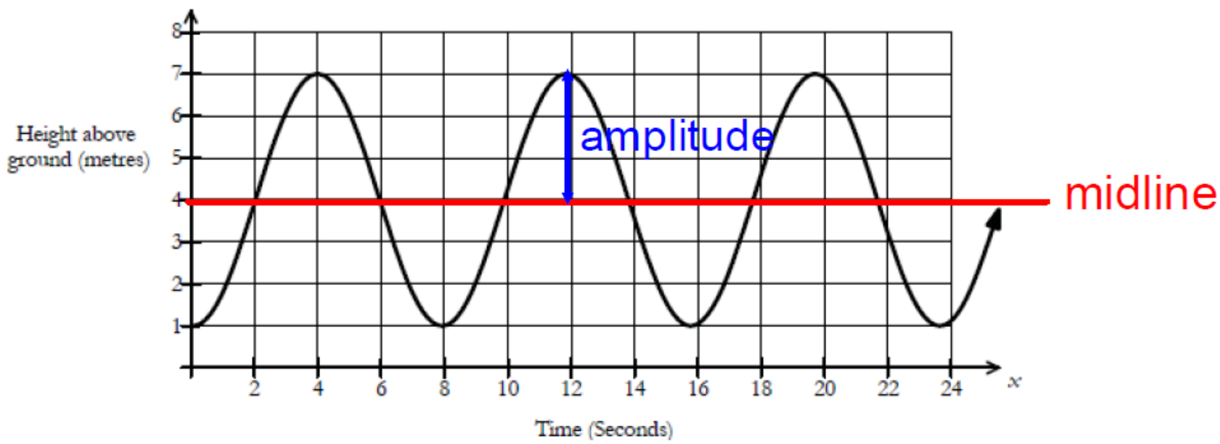
$C = 90^o$ translates 90^o right

(B) $y = 0.6\sin 2(x + 45^o) - 3$

$C = -45^o$ translates 45^o left

Maximum/Minimum Points

Midline can be read from a graph, or it can be calculated if we are given an equation.



Maximum Point $=$ midline $+$ amplitude

Minimum Point $=$ midline $-$ amplitude

Recall:

amplitude = a and the equation of the midline is $y = d$

Thus,

$$\text{Maximum Point} = \text{midline} + \text{amplitude} = \text{d} + \text{a}$$

$$\text{Minimum Point} = \text{midline} - \text{amplitude} = \text{d} - \text{a}$$

$$\text{Maximum Point} = \text{d} + \text{a}$$

$$\text{Minimum Point} = \text{d} - \text{a}$$

Domain and Range of a Sinusoidal Equation

For a sinusoidal function the domain is as follows:

$$\{x | x \in \text{R}\}$$

This may change in application problems in which the x-values are restricted.

For a sinusoidal function the range is as follows:

$$\{y|\text{minimum} \le y \le \text{maximum}, y \in \mathrm{R}\}$$

Example 8:
State the domain and range for the following sinusoidal functions.

(A) $y = \overset{a}{2}\sin\overset{b}{3}(x \overset{c}{-90^{o}}) \overset{d}{+1}$ D: $\{x | x \in R\}$

max $= d + a = 1 + 2 = 3$

min $= d - a = 1 - 2 = -1$ R: $\{y | -1 \le y \le 3, y \in R\}$

(B) $y = \overset{a}{4}\cos\overset{b}{2}(x \overset{c}{+ 60^{o}}) \overset{d}{- 3}$ D: $\{x | x \in R\}$

max $= -3 + 4 = 1$

min $= -3 - 4 = -7$ R: $\{y | -7 \le y \le 1, y \in R\}$

Example 9:
Consider the function:

$$y = \overset{a}{2}\cos\overset{b}{4}x \overset{d}{+ 1}$$

Describe the function by stating the amplitude, equation of the midline, range, and period, as well as the relevant horizontal translation of $y = \cos x$

amplitude: $a = 2$

midline: $d = 1 \therefore y = 1$

max $= d + a = 1 + 2 = 3$, min $= 1 - 2 = -1$

R: $\{y | -1 \le y \le 3, y \in R\}$

period $= \dfrac{360^{o}}{4} = 90^{o}$

$C = 0^{o}$ HT: 0^{o}

Example 10:
Consider the function:

$$y = \underset{a}{3}\sin\underset{b}{2}(x \underset{c}{- 45^{o}}) \underset{d}{\ }$$

Describe the function by stating the amplitude, equation of the midline, range, and period, as well as the relevant horizontal translation of $y = \sin x$

amplitude: $a = 3$

midline: $d = 0 \therefore y = 0$

max $= 0 + 3 = 3$

min $= 0 - 3 = -3$

R: $\{y | -3 \le y \le 3, y \in R\}$

period $= \dfrac{360^{o}}{2} = 180^{o}$

$C = 45^{o}$ HT: 45^{o} right

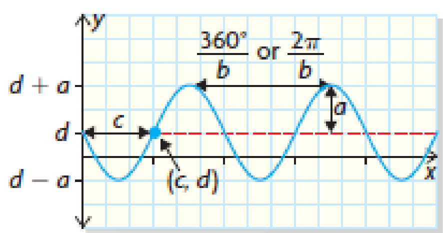
In Summary

Key Idea

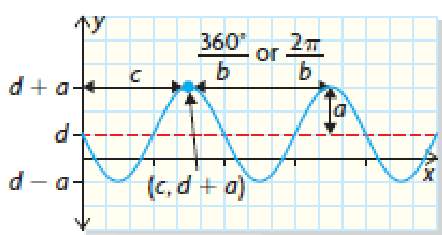
- Any sinusoidal function can be expressed as either a cosine function or a sine function.

Need to Know

- A sinusoidal function of the form
$$y = a \sin b(x - c) + d \text{ or}$$
$$y = a \cos b(x - c) + d$$
has the following characteristics:
  - The value of a is the **amplitude:**
  $$a = \frac{\text{maximum value} - \text{minimum value}}{2}$$
  - The value of b is the number of cycles in 360° or 2π. The **period** is $\frac{360^\circ}{b}$ or $\frac{2\pi}{b}$.
  - The value of c indicates the horizontal translation that has been applied to the graph of $y = \sin x$ or $y = \cos x$.
  - The **equation of the midline** is
  $$y = d$$
  where
  $$d = \frac{\text{maximum value} + \text{minimum value}}{2}$$
  - The **maximum value** is $d + a$, and the **minimum value** is $d - a$.
- In the graph of a sine function, c is the distance from the vertical axis to the first midline point where the function is increasing.



- In the graph of a cosine function, c is the distance from the vertical axis to the first maximum point.



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